Chapter 8

Conic Sections

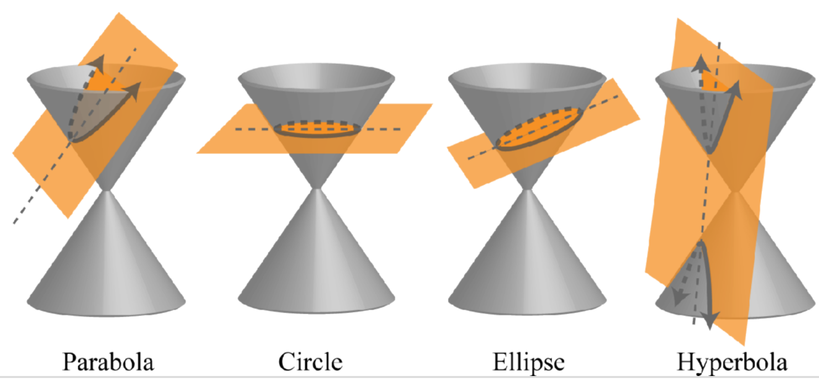
8.1 Distance, Midpoint, and the Parabola

LEARNING OBJECTIVES

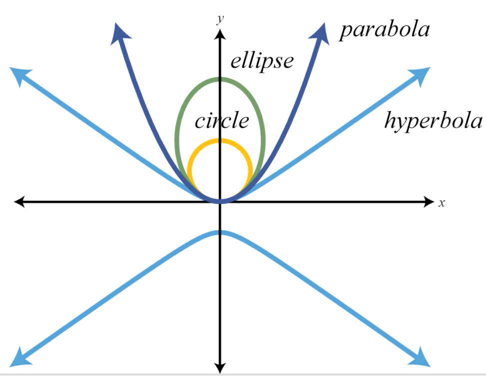
1. Apply the distance and midpoint formulas.
2. Graph a parabola using its equation given in standard from.
3. Determine standard form for the equation of a parabola given general form.

Conic Sections

A **conic section**[1] is a curve obtained from the intersection of a right circular cone and a plane. The conic sections are the parabola, circle, ellipse, and hyperbola.



The goal is to sketch these graphs on a rectangular coordinate plane.



1. A curve obtained from the intersection of a right circular cone and a plane.

The Distance and Midpoint Formulas

We begin with a review of the **distance formula**[2]. Given two points (x_1, y_1) and (x_2, y_2) in a rectangular coordinate plane, the distance d between them is given by the distance formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Furthermore, the point that bisects the line segment formed by these two points is called the **midpoint**[3] and is given by the formula,

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The midpoint is an ordered pair formed by the average of the x-values and the average of the y-values.

2. Given two points (x_1, y_1) and (x_2, y_2), the distance d between them is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

3. Given two points (x_1, y_1) and (x_2, y_2), the midpoint is an ordered pair given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Example 1

Given $\left(-2, -5\right)$ and $(-4, -3)$ calculate the distance and midpoint between them.

Solution:

In this case, we will use the formulas with the following points:

$$\begin{array}{ll} \left(x_1,\ y_1\right) & \left(x_2,\ y_2\right) \\ \left(-2, -5\right) & (-4, -3) \end{array}$$

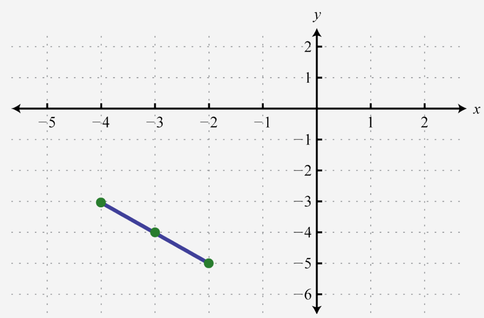
It is a good practice to include the formula in its general form before substituting values for the variables; this improves readability and reduces the probability of making errors.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + \left(y_2 - y_1\right)^2} \\ &= \sqrt{\left[-4 - \ (-2)\right]^2 + \left[-3 - \ (-5)\right]^2} \\ &= \sqrt{(-4 + 2)^2 + \left(-3 + 5\right)^2} \\ &= \sqrt{(-2)^2 + (2)^2} \\ &= \sqrt{4 + 4} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

Next determine the midpoint.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-2 + (-4)}{2}, \frac{-5 + (-3)}{2}\right)$$
$$= \left(\frac{-6}{2}, \frac{-8}{2}\right)$$
$$= (-3, -4)$$

Plotting these points on a graph we have,



Answer: Distance: $2\sqrt{2}$ units; midpoint: $(-3, -4)$

Example 2

The diameter of a circle is defined by the two points $(-1, 2)$ and $(1, -2)$. Determine the radius of the circle and use it to calculate its area.

Solution:

Find the diameter using the distance formula.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + \left(y_2 - y_1\right)^2} \\ &= \sqrt{\left[1 - (-1)\right]^2 + (-2 - 2)^2} \\ &= \sqrt{(2)^2 + (-4)^2} \\ &= \sqrt{4 + 16} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

Recall that the radius of a circle is one-half of the circle's diameter. Therefore, if $d = 2\sqrt{5}$ units, then

$$r = \frac{d}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$$

The area of a circle is given by the formula $A = \pi r^2$ and we have

$$\begin{aligned} A &= \pi\left(\sqrt{5}\right)^2 \\ &= \pi \cdot 5 \\ &= 5\pi \end{aligned}$$

Area is measured in square units.

Answer: Radius: $\sqrt{5}$ units; area: 5π square units

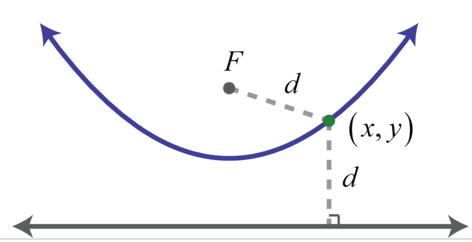
Try this! Given $(0, 0)$ and $(9, -3)$ calculate the distance and midpoint between them.

Answer: Distance: $3\sqrt{10}$ units; midpoint: $\left(\frac{9}{2}, -\frac{3}{2}\right)$

(click to see video)

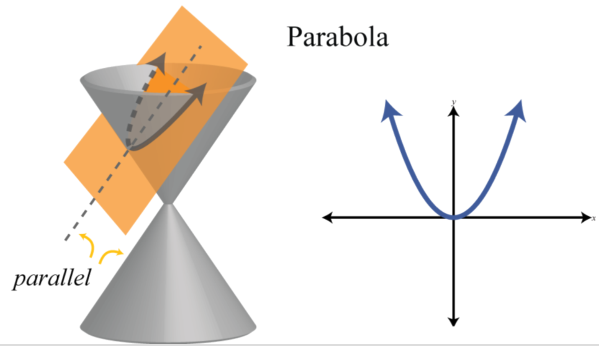
The Parabola

A **parabola**[4] is the set of points in a plane equidistant from a given line, called the directrix, and a point not on the line, called the focus. In other words, if given a line L the directrix, and a point F the focus, then (x, y) is a point on the parabola if the shortest distance from it to the focus and from it to the line is equal as pictured below:



The vertex of the parabola is the point where the shortest distance to the directrix is at a minimum. In addition, a parabola is formed by the intersection of a cone with an oblique plane that is parallel to the side of the cone:

4. The set of points in a plane equidistant from a given line, called the directrix, and a point not on the line, called the focus.



Recall that the graph of a quadratic function, a polynomial function of degree 2, is parabolic. We can write the equation of a **parabola in general form**[5] or we can write the equation of a **parabola in standard form**[6]:

$$\begin{array}{cc} \textit{General Form} & \textit{Standard Form} \\ y = ax^2 + bx + c & y = a(x-h)^2 + k \end{array}$$

Here a, b, and c are real numbers, $a \neq 0$. Both forms are useful in determining the general shape of the graph. However, in this section we will focus on obtaining standard form, which is often called **vertex form**[7]. Given a quadratic function in standard form, the vertex is (h, k) . To see that this is the case, consider graphing $y = (x+3)^2 + 2$ using transformations.

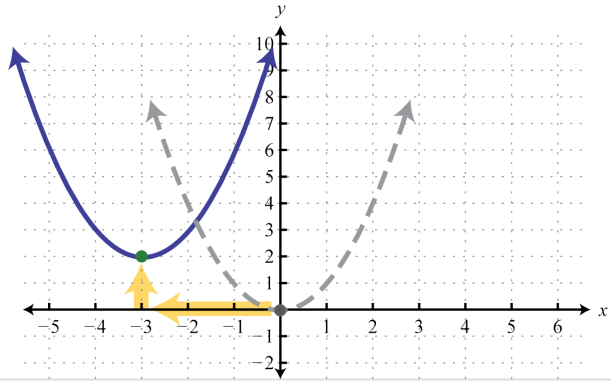
5. The equation of a parabola written in the form $y = ax^2 + bx + c$ or $x = ay^2 + by + c$, where a, b, and c are real numbers and $a \neq 0$.

6. The equation of a parabola written in the form $y = a(x-h)^2 + k$ or $x = a(y-k)^2 + h$.

7. The equation of a parabola written in standard form is often called vertex form. In this form the vertex is apparent: (h, k) .

$$\begin{array}{ll} y = x^2 & \textit{Basic squaring function.} \\ y = (x+3)^2 & \textit{Horizontal shift left 3 units.} \\ y = (x+3)^2 + 2 & \textit{Vertical shift up 2 units.} \end{array}$$

Use these translations to sketch the graph,



Here we can see that the vertex is $(-3, 2)$. This can be determined directly from the equation in standard form,

$$\begin{array}{l} y = \quad a(x - \underset{\downarrow}{h})^2 + \underset{\downarrow}{k} \\ y = [x - \quad (-3)]^2 + 2 \end{array}$$

Written in this form we can see that the vertex is $(-3, 2)$. However, the equation is typically not given in standard form. Transforming general form to standard form, by completing the square, is the main process by which we will sketch all of the conic sections.

Example 3

Rewrite the equation in standard form and determine the vertex of its graph: $y = x^2 - 8x + 15$.

Solution:

Begin by making room for the constant term that completes the square.

$$\begin{aligned} y &= x^2 - 8x + 15 \\ &= x^2 - 8x + \_\_\_ + 15 - \_\_\_ \end{aligned}$$

The idea is to add and subtract the value that completes the square, $\left(\frac{b}{2}\right)^2$, and then factor. In this case, add and subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$.

$$\begin{aligned} y &= x^2 - 8x + 15 && \textit{Add and subtract 16.} \\ &= \left(x^2 - 8x + 16\right) + 15 - 16 && \textit{Factor.} \\ &= (x-4)\,(x-4) - 1 \\ &= (x-4)^2 - 1 \end{aligned}$$

Adding and subtracting the same value within an expression does not change it. Doing so is equivalent to adding 0. Once the equation is in this form, we can easily determine the vertex.

$$\begin{aligned} y &= a(x - h)^2 + \quad k \\ & \qquad\quad \downarrow \qquad\ \ \downarrow \\ y &= \quad (x - 4)^2 + (-1) \end{aligned}$$

Here we have a translation to the right 4 units and down 1 unit. Hence, $h = 4$ and $k = -1$.

Answer: $y = (x - 4)^2 - 1$; vertex: $(4, -1)$

If there is a leading coefficient other than 1, then begin by factoring out that leading coefficient from the first two terms of the trinomial.

Example 4

Rewrite the equation in standard form and determine the vertex of the graph: $y = -2x^2 + 12x - 16$.

Solution:

Since $a = -2$, factor this out of the first two terms in order to complete the square. Leave room inside the parentheses to add and subtract the value that completes the square.

$$\begin{aligned} y &= -2x^2 + 12x - 16 \\ &= -2\left(x^2 - 6x + \_\_\_ - \_\_\_\right) - 16 \end{aligned}$$

Now use -6 to determine the value that completes the square. In this case, $\left(\frac{b}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$. Add and subtract 9 and factor as follows:

$$\begin{aligned} y &= -2x^2 + 12x - 16 \\ &= -2\left(x^2 - 6x + \_\_\_ - \_\_\_\right) - 16 && \textit{Add and subtract 9.} \\ &= -2(x^2 - 6x + 9 - 9) - 16 && \textit{Factor.} \\ &= -2\left[(x - 3)(x - 3) - 9\right] - 16 \\ &= -2\left[(x - 3)^2 - 9\right] - 16 && \textit{Distribute the } -2. \\ &= -2(x - 3)^2 + 18 - 16 \\ &= -2(x - 3)^2 + 2 \end{aligned}$$

In this form, we can easily determine the vertex.

$$\begin{aligned} y &= \ a(x-h)^2 + k \\ & \qquad\qquad\ \ \downarrow \quad \downarrow \\ y &= -2(x-3)^2 + 2 \end{aligned}$$

Here $h = 3$ and $k = 2$.

Answer: $y = -2(x-3)^2 + 2$; vertex: $(3, 2)$

Make use of both general form and standard form when sketching the graph of a parabola.

Example 5

Graph: $y = -2x^2 + 12x - 16$.

Solution:

From the previous example we have two equivalent forms of this equation,

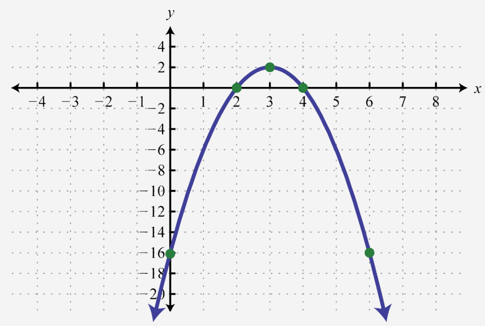
$$\begin{array}{cc} \textit{General Form} & \textit{Standard Form} \\ y = -2x^2 + 12x - 16 & y = -2(x-3)^2 + 2 \end{array}$$

Recall that if the leading coefficient $a > 0$ the parabola opens upward and if $a < 0$ the parabola opens downward. In this case, $a = -2$ and we conclude the parabola opens downward. Use general form to determine the *y*-intercept. When $x = 0$ we can see that the *y*-intercept is $(0, -16)$. From the equation in standard form, we can see that the vertex is $(3, 2)$. To find the *x*-intercept we could use either form. In this case, we will use standard form to determine the *x*-values where $y = 0$,

$$\begin{aligned} y &= -2(x-3)^2 + 2 && \textit{Set } y = 0 \textit{ and solve.} \\ 0 &= -2(x-3)^2 + 2 \\ -2 &= -2(x-3)^2 \\ 1 &= (x-3)^2 && \textit{Apply the square root property.} \\ \pm 1 &= x - 3 \\ 3 \pm 1 &= x \end{aligned}$$

Here $x = 3 - 1 = 2$ or $x = 3 + 1 = 4$ and therefore the *x*-intercepts are $(2, 0)$ and $(4, 0)$. Use this information to sketch the graph.

Answer:



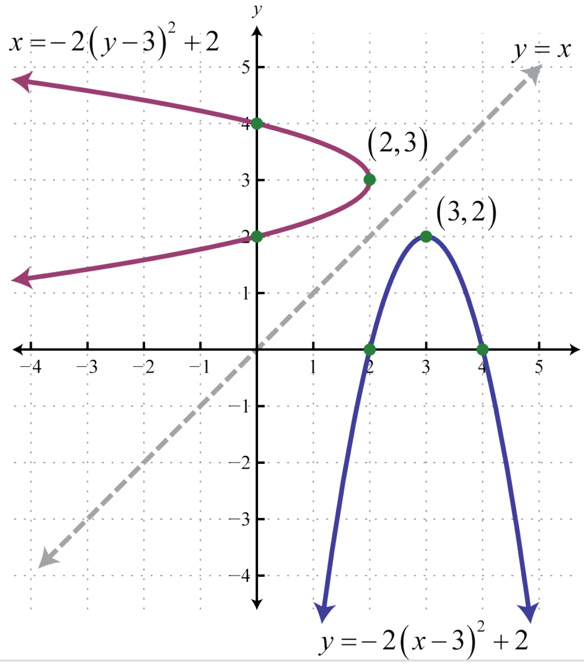
So far we have been sketching parabolas that open upward or downward because these graphs represent functions. At this point we extend our study to include parabolas that open right or left. If we take the equation that defines the parabola in the previous example,

$$y = -2(x-3)^2 + 2$$

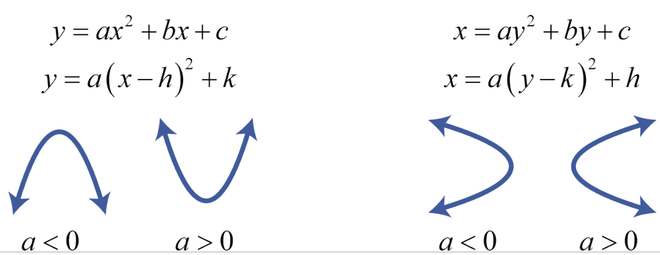
and switch the x and y values we obtain

$$x = -2(y-3)^2 + 2$$

This produces a new graph with symmetry about the line $y = x$.



Note that the resulting graph is not a function. However, it does have the same general parabolic shape that opens left. We can recognize equations of parabolas that open left or right by noticing that they are quadratic in *y* instead of *x*. Graphing parabolas that open left or right is similar to graphing parabolas that open upward and downward. In general, we have



In all cases, the vertex is (h, k) . Take care to note the placement of *h* and *k* in each equation.

Example 6

Graph: $x = y^2 + 10y + 13$.

Solution:

Because the coefficient of y^2 is positive, $a = 1$, we conclude that the graph is a parabola that opens to the right. Furthermore, when $y = 0$ it is clear that $x = 13$ and therefore the x-intercept is $(13, 0)$. Complete the square to obtain standard form. Here we will add and subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2 = (5)^2 = 25$.

$$\begin{aligned} x &= y^2 + 10y + 13 \\ &= y^2 + 10y + 25 - 25 + 13 \\ &= (y + 5)\ (y + 5) - 12 \\ &= (y + 5)^2 - 12 \end{aligned}$$

Therefore,

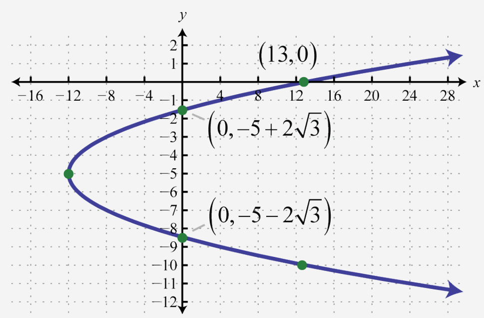
$$\begin{array}{c} x = \ a\ (y - k)^2 \ + \ h \\ \qquad\quad \downarrow \qquad\ \ \downarrow \\ x = (y - \ (-5))^2 + (-12) \end{array}$$

From this we can see that the vertex $(h, k) = \ (-12, -5)$ .Next use standard form to find the y-intercepts by setting $x = 0$.

$$\begin{aligned} x &= (y+5)^2 - 12 \\ 0 &= (y+5)^2 - 12 \\ 12 &= (y+5)^2 \\ \pm\sqrt{12} &= y+5 \\ \pm 2\sqrt{3} &= y+5 \\ -5 \pm 2\sqrt{3} &= y \end{aligned}$$

The y-intercepts are $\left(0, -5 - 2\sqrt{3}\right)$ and $\left(0, -5 + 2\sqrt{3}\right)$. Use this information to sketch the graph.

Answer:



Example 7

Graph: $x = -2y^2 + 4y - 5$.

Solution:

Because the coefficient of y^2 is $a = -2$, we conclude that the graph is a parabola that opens to the left. Furthermore, when $y = 0$ it is clear that $x = -5$ and therefore the x-intercept is $(-5, 0)$. Begin by factoring out the leading coefficient as follows:

$$\begin{aligned} x &= -2y^2 + 4y - 5 \\ &= -2\left(y^2 - 2y + \_\_\_ - \_\_\_\right) - 5 \end{aligned}$$

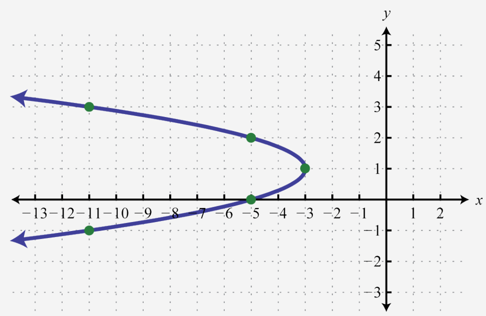
Here we will add and subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$.

$$\begin{aligned} x &= -2y^2 + 4y - 5 \\ &= -2(y^2 - 2y + 1 - 1) - 5 \\ &= -2\left[(y-1)^2 - 1\right] - 5 \\ &= -2(y-1)^2 + 2 - 5 \\ &= -2(y-1)^2 - 3 \end{aligned}$$

Therefore, from vertex form, $x = -2(y-1)^2 - 3$, we can see that the vertex is $(h, k) = (-3, 1)$. Because the vertex is at $(-3, 1)$ and the parabola opens to the left, we can conclude that there are no y-intercepts. Since we only have two points, choose some y-values and find the corresponding x-values.

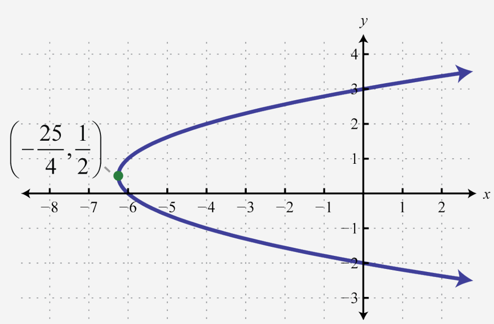
x	y	$x = -2(y-1)^2 - 3$
-11	-1	$x = -2(-1-1)^2 - 3 = -2(-2)^2 - 3 = -11$
-5	2	$x = -2(2-1)^2 - 3 = -2(1)^2 - 3 = -5$
-11	3	$x = -2(3-1)^2 - 3 = -2(2)^2 - 3 = -11$

Answer:



Try this! Graph: $x = y^2 - y - 6$.

Answer:



(click to see video)

KEY TAKEAWAYS

- Use the distance formula to determine the distance between any two given points. Use the midpoint formula to determine the midpoint between any two given points.
- A parabola can open upward or downward, in which case, it is a function. In this section, we extend our study of parabolas to include those that open left or right. Such graphs do not represent functions.
- The equation of a parabola that opens upward or downward is quadratic in *x*, $y = ax^2 + bx + c$. If $a > 0$, then the parabola opens upward and if $a < 0$, then the parabola opens downward.
- The equation of a parabola that opens left or right is quadratic in *y*, $x = ay^2 + by + c$. If $a > 0$, then the parabola opens to the right and if $a < 0$, then the parabola opens to the left.
- The equation of a parabola in general form $y = ax^2 + bx + c$ or $x = ay^2 + by + c$ can be transformed to standard form $y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$ by completing the square.
- When completing the square, ensure that the leading coefficient of the variable grouping is 1 before adding and subtracting the value that completes the square.
- Both general and standard forms are useful when graphing parabolas. Given standard form the vertex is apparent (h, k). To find the *x*-intercept set $y = 0$ and solve for *x* and to find the *y*-intercept set $x = 0$ and solve for *y*.

TOPIC EXERCISES

PART A: THE DISTANCE AND MIDPOINT FORMULAS

Calculate the distance and midpoint between the given two points.

1. $(-1, -3)$ and $(5, -11)$
2. $(-3, 2)$ and $(1, -1)$
3. $(4, -2)$ and $(-2, -6)$
4. $(-5, -6)$ and $(-3, -4)$
5. $(10, -1)$ and $(9, 6)$
6. $(-6, -4)$ and $(-12, 1)$
7. $(0, 0)$ and $(\sqrt{2}, \sqrt{3})$
8. $(0, 0)$ and $(2\sqrt{2}, -\sqrt{3})$
9. $(\sqrt{5}, -\sqrt{3})$ and $(2\sqrt{5}, -\sqrt{3})$
10. $(3\sqrt{10}, \sqrt{6})$ and $(\sqrt{10}, -5\sqrt{6})$
11. $\left(\frac{1}{2}, -1\right)$ and $\left(-2, \frac{3}{2}\right)$
12. $\left(-\frac{4}{3}, 2\right)$ and $\left(-\frac{1}{3}, -\frac{1}{2}\right)$
13. $\left(\frac{1}{5}, -\frac{9}{5}\right)$ and $\left(\frac{3}{10}, -\frac{5}{2}\right)$
14. $\left(-\frac{1}{2}, \frac{4}{3}\right)$ and $\left(-\frac{2}{3}, \frac{5}{6}\right)$
15. (a, b) and $(0, 0)$
16. $(0, 0)$ and $(a\sqrt{2}, 2\sqrt{a})$

Determine the area of a circle whose diameter is defined by the given two points.

17. $(-8, 12)$ and $(-6, 8)$
18. $(9, 5)$ and $(9, -1)$
19. $(7, -8)$ and $(5, -10)$
20. $(0, -5)$ and $(6, 1)$
21. $\left(\sqrt{6}, 0\right)$ and $\left(0, 2\sqrt{3}\right)$
22. $\left(0, \sqrt{7}\right)$ and $\left(\sqrt{5}, 0\right)$

Determine the perimeter of the triangle given the coordinates of the vertices.

23. $(5, 3)$, $(2, -3)$, and $(8, -3)$
24. $(-3, 2)$, $(-4, -1)$, and $(-1, 0)$
25. $(3, 3)$, $\left(5, 3 - 2\sqrt{3}\right)$, and $(7, 3)$
26. $(0, 0)$, $\left(0, 2\sqrt{2}\right)$, and $\left(\sqrt{2}, 0\right)$

Find *a* so that the distance *d* between the points is equal to the given quantity.

27. $(1, 2)$ and $(4, a)$; $d = 5$ units
28. $(-3, a)$ and $(5, 6)$; $d = 10$ units
29. $(3, 1)$ and $(a, 0)$; $d = \sqrt{2}$ units
30. $(a, 1)$ and $(5, 3)$; $d = \sqrt{13}$ units

PART B: THE PARABOLA

Graph. Be sure to find the vertex and all intercepts.

31. $y = x^2 + 3$

32. $y = \frac{1}{2}(x - 4)^2$

33. $y = -2(x + 1)^2 - 1$

34. $y = -(x - 2)^2 + 1$

35. $y = -x^2 + 3$

36. $y = -(x + 1)^2 + 5$

37. $x = y^2 + 1$

38. $x = y^2 - 4$

39. $x = \left(y + 2\right)^2$

40. $x = \left(y - 3\right)^2$

41. $x = -y^2 + 2$

42. $x = -\left(y + 1\right)^2$

43. $x = \frac{1}{3}\left(y - 3\right)^2 - 1$

44. $x = -\frac{1}{3}\left(y + 3\right)^2 - 1$

Rewrite in standard form and give the vertex.

45. $y = x^2 - 6x + 18$

46. $y = x^2 + 8x + 36$

47. $x = y^2 + 20y + 87$

48. $x = y^2 - 10y + 21$

49. $y = x^2 - 14x + 49$

50. $x = y^2 + 16y + 64$

51. $x = 2y^2 - 4y + 5$

52. $y = 3x^2 - 30x + 67$

53. $y = 6x^2 + 36x + 54$

54. $x = 3y^2 + 6y - 1$

55. $y = 2x^2 - 2x - 1$

56. $x = 5y^2 + 15y + 9$

57. $x = -y^2 + 5y - 5$

58. $y = -x^2 + 9x - 20$

Rewrite in standard form and graph. Be sure to find the vertex and all intercepts.

59. $y = x^2 - 4x - 5$

60. $y = x^2 + 6x - 16$

61. $y = -x^2 + 12x - 32$

62. $y = -x^2 - 10x$

63. $y = 2x^2 + 4x + 9$

64. $y = 3x^2 - 6x + 4$

65. $y = -5x^2 + 30x - 45$

66. $y = -4x^2 - 16x - 16$

67. $x = y^2 - 2y - 8$

68. $x = y^2 + 4y + 8$

69. $x = y^2 - 2y - 3$

70. $x = y^2 + 6y - 7$

71. $x = -y^2 - 10y - 24$

72. $x = -y^2 - 12y - 40$

73. $x = 3y^2 + 12y + 12$

74. $x = -2y^2 + 12y - 18$

75. $x = y^2 - 4y - 3$

76. $x = y^2 + 6y + 1$

77. $x = -y^2 + 2y + 5$

78. $y = 2x^2 - 2x + 1$

79. $y = -3x^2 + 2x + 1$

80. $y = -x^2 + 3x + 10$

81. $x = -4y^2 - 4y - 5$

82. $x = y^2 - y + 2$

83. $y = x^2 + 5x - 1$

84. $y = 2x^2 + 6x + 3$

85. $x = 2y^2 + 10x + 12$

86. $x = y^2 + y - 1$

PART C: DISCUSSION BOARD

87. Research and discuss real-world applications that involve a parabola.

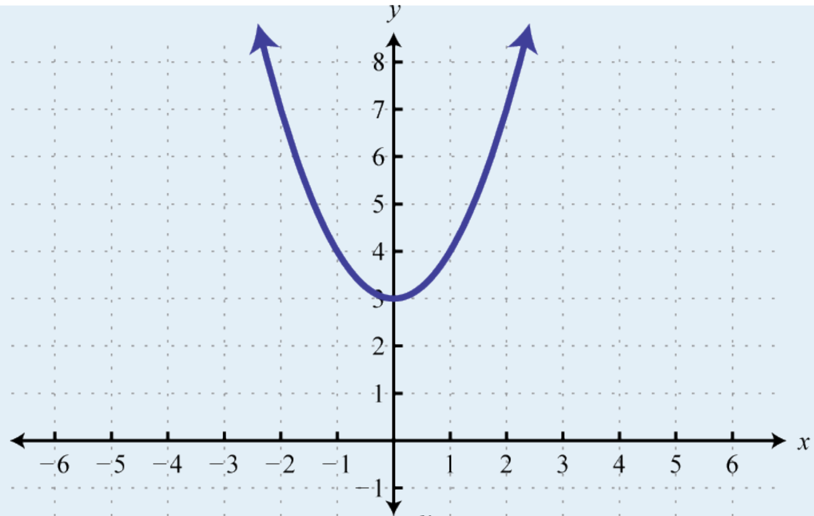
88. Do all parabolas have *x*-intercepts? Explain.

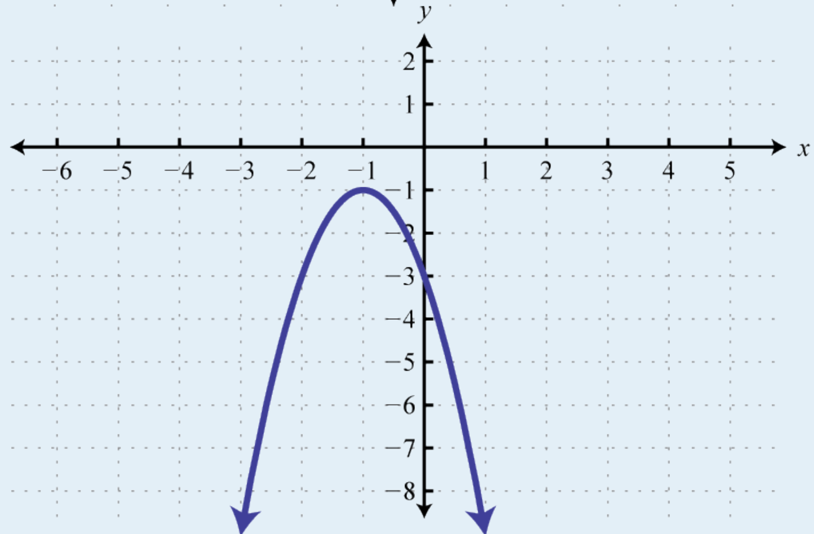
89. Do all parabolas have *y*-intercepts? Explain.

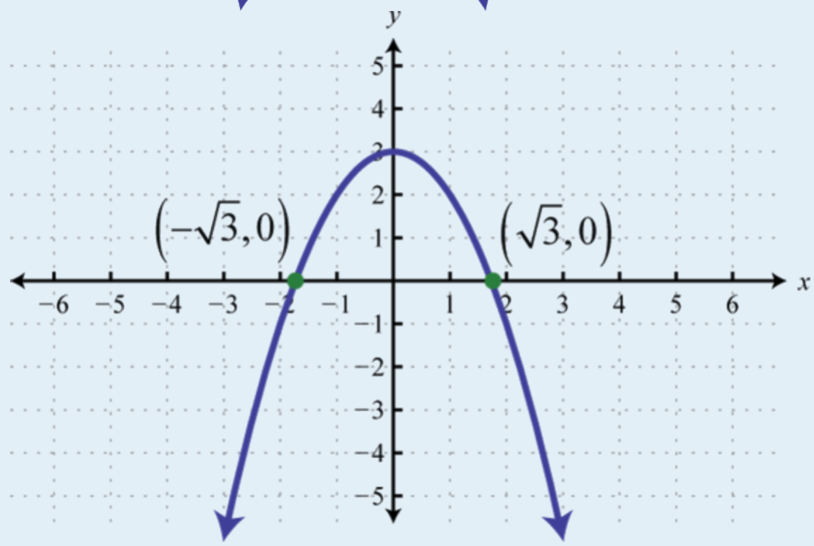
90. Make up your own parabola that opens left or right, write it in general form, and graph it.

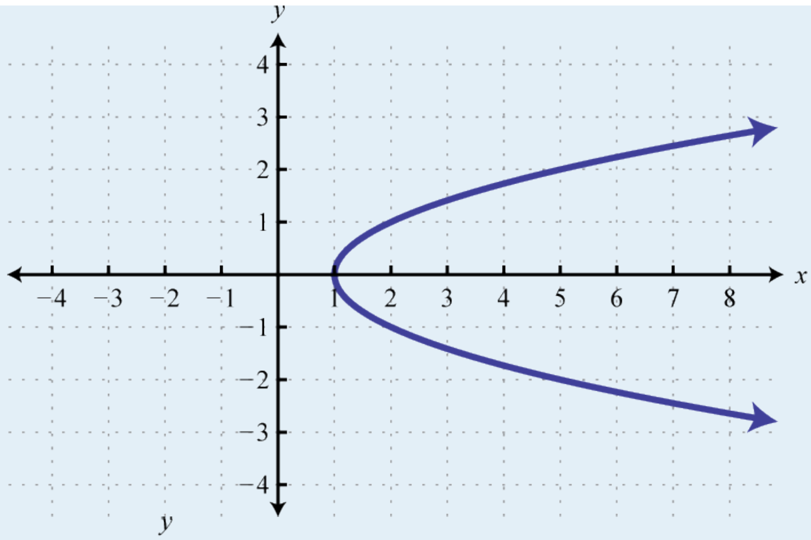
ANSWERS

1. Distance: 10 units; midpoint: $(2, -7)$
3. Distance: $2\sqrt{13}$ units; midpoint: $(1, -4)$
5. Distance: $5\sqrt{2}$ units; midpoint: $\left(\frac{19}{2}, \frac{5}{2}\right)$
7. Distance: $\sqrt{5}$ units; midpoint: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right)$
9. Distance: $\sqrt{5}$ units; midpoint: $\left(\frac{3\sqrt{5}}{2}, -\sqrt{3}\right)$
11. Distance: $\frac{5\sqrt{2}}{2}$ units; midpoint: $\left(-\frac{3}{4}, \frac{1}{4}\right)$
13. Distance: $\frac{\sqrt{2}}{2}$ units; midpoint: $\left(\frac{1}{4}, -\frac{43}{20}\right)$
15. Distance: $\sqrt{a^2 + b^2}$ units; midpoint: $\left(\frac{a}{2}, \frac{b}{2}\right)$
17. 5π square units
19. 2π square units
21. $\frac{9}{2}\pi$ square units
23. $6 + 6\sqrt{5}$ units
25. 12 units
27. −2, 6
29. 2, 4

31. 

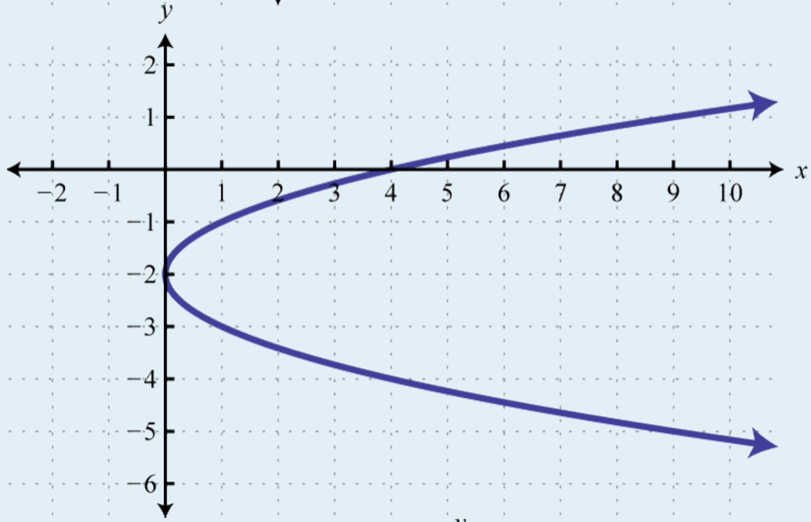
33. 

35. 



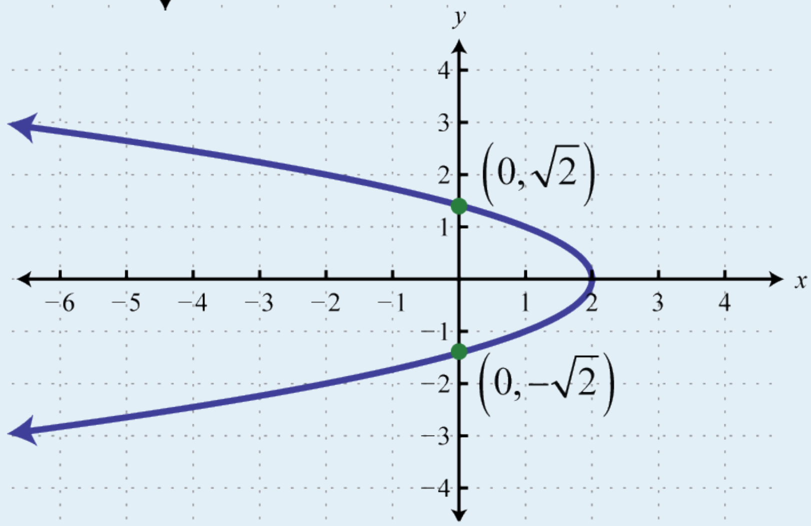
y
x
4
3
2
1
−4
−3
−2
−1
1
2
3
4
5
6
7
8
−1
−2
−3
−4

37.



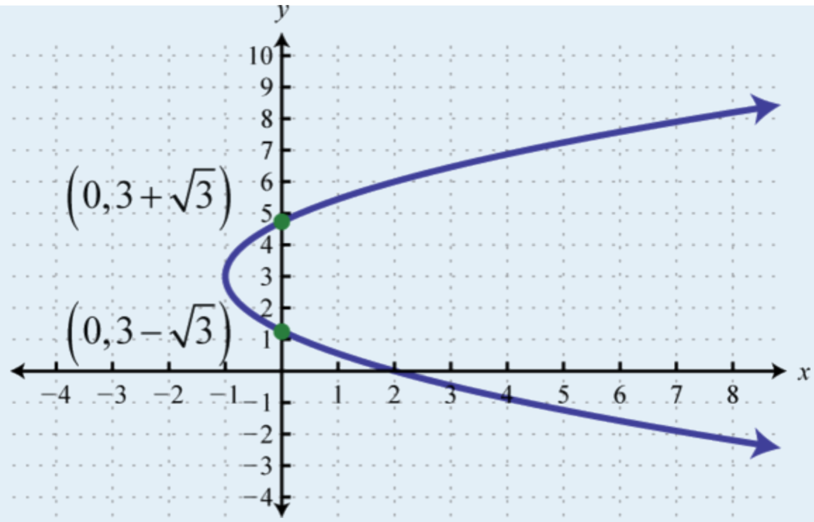
y
x
2
1
−2
−1
1
2
3
4
5
6
7
8
9
10
−1
−2
−3
−4
−5
−6

39.



y
x
4
3
2
1
$(0, \sqrt{2})$
−6
−5
−4
−3
−2
−1
1
2
3
4
−1
−2
$(0, -\sqrt{2})$
−3
−4

41.

43. 

45. $y = (x-3)^2 + 9$; vertex: $(3, 9)$

47. $x = \left(y+10\right)^2 - 13$; vertex: $(-13, -10)$

49. $y = (x-7)^2$; vertex: $(7, 0)$

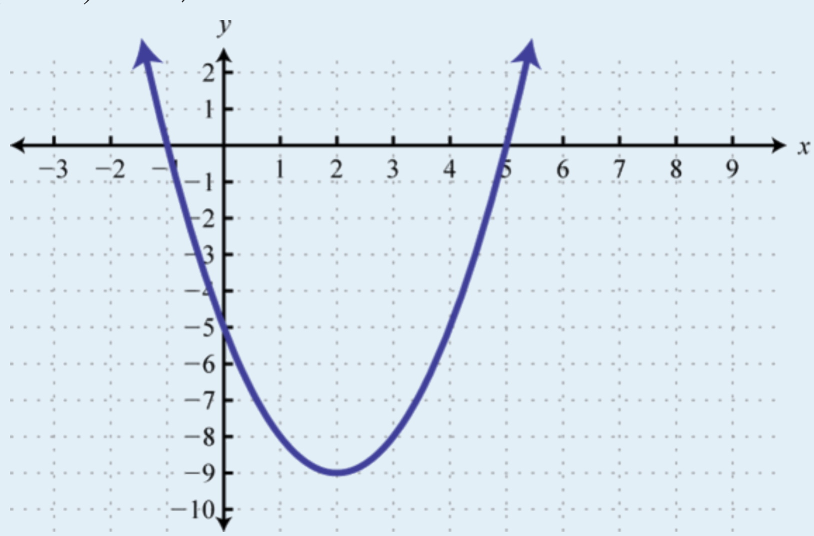
51. $x = 2\left(y-1\right)^2 + 3$; vertex: $(3, 1)$

53. $y = 6(x+3)^2$; vertex: $(-3, 0)$

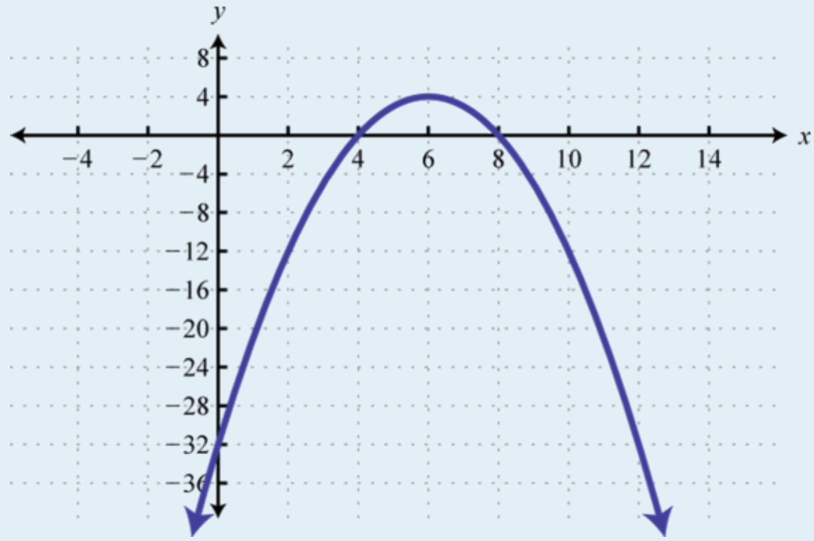
55. $y = 2\left(x - \frac{1}{2}\right)^2 - \frac{3}{2}$; vertex: $\left(\frac{1}{2}, -\frac{3}{2}\right)$

57. $x = -\left(y - \frac{5}{2}\right)^2 + \frac{5}{4}$; vertex: $\left(\frac{5}{4}, \frac{5}{2}\right)$

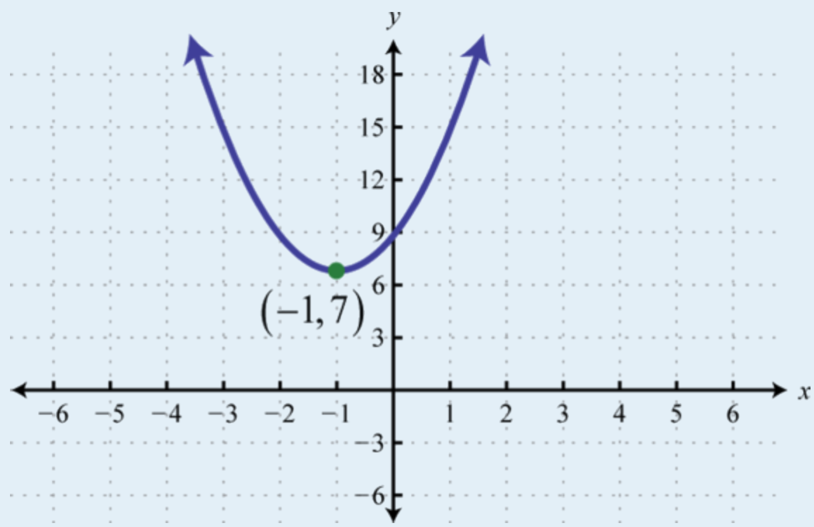
59. $y = (x-2)^2 - 9$;



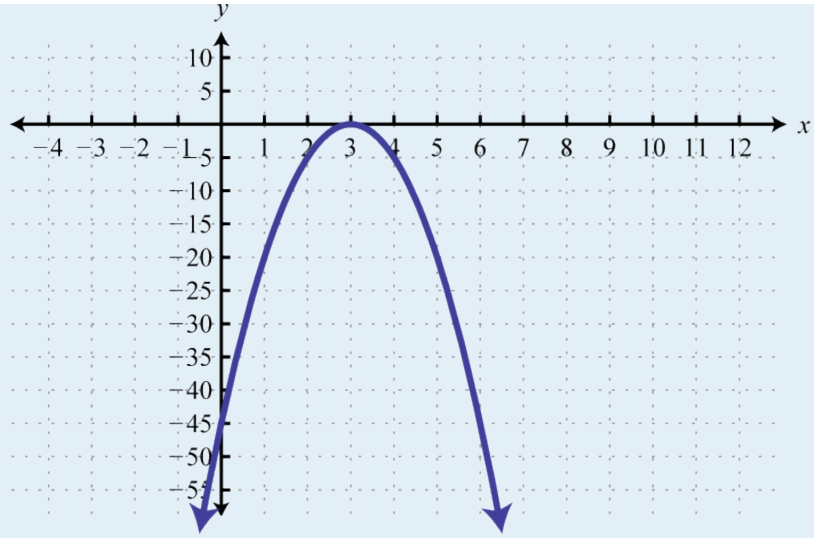
61. $y = -(x-6)^2 + 4$;



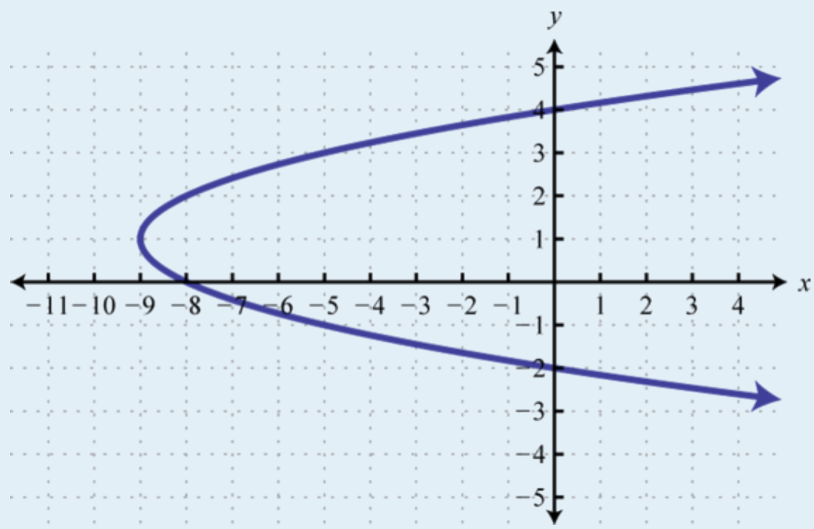
63. $y = 2(x+1)^2 + 7$;



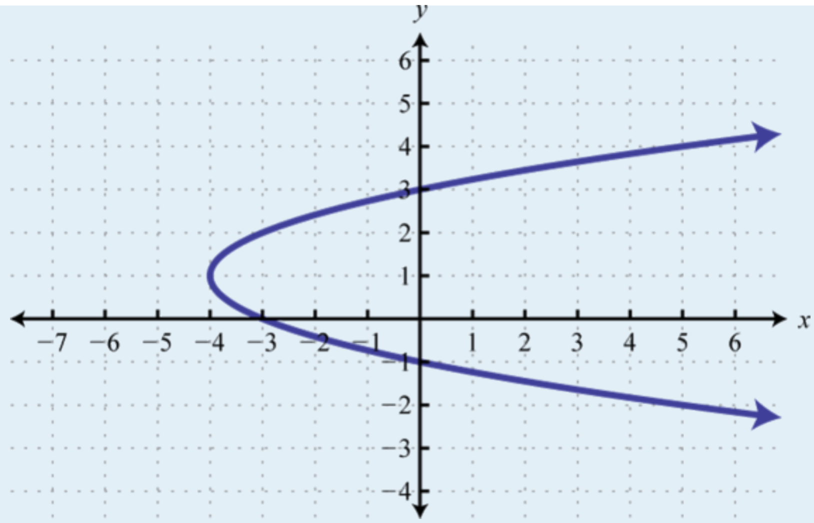
65. $y = -5(x-3)^2$;



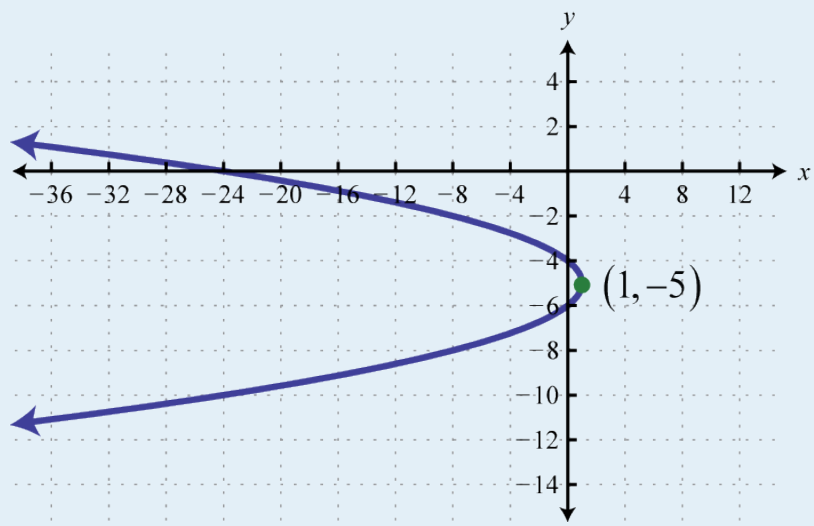
67. $x = (y-1)^2 - 9$;



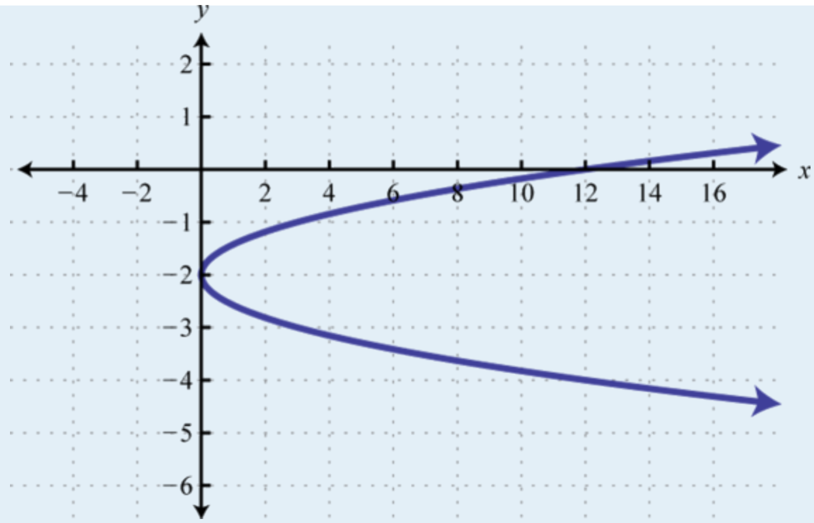
69. $x = (y-1)^2 - 4$;



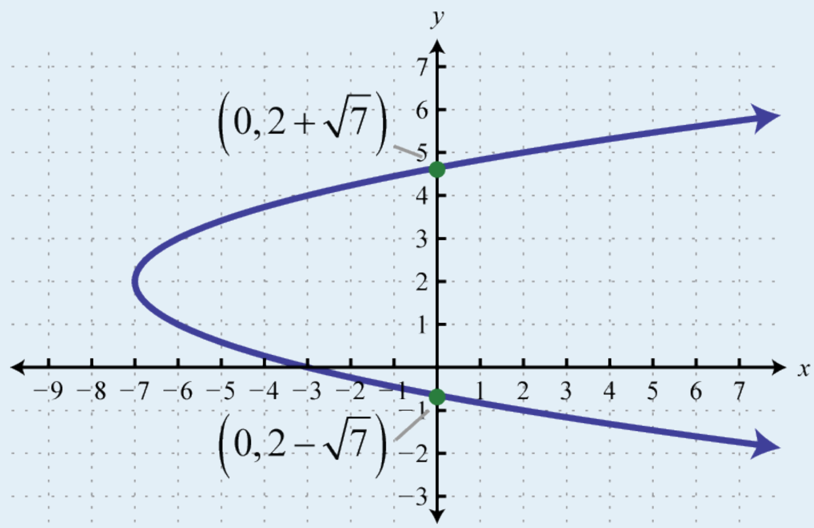
71. $x = -(y+5)^2 + 1$;



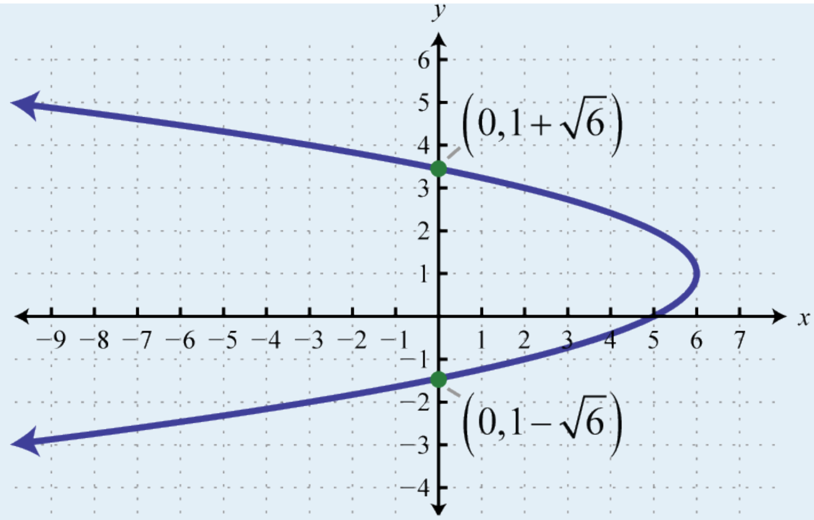
73. $x = 3(y+2)^2$;



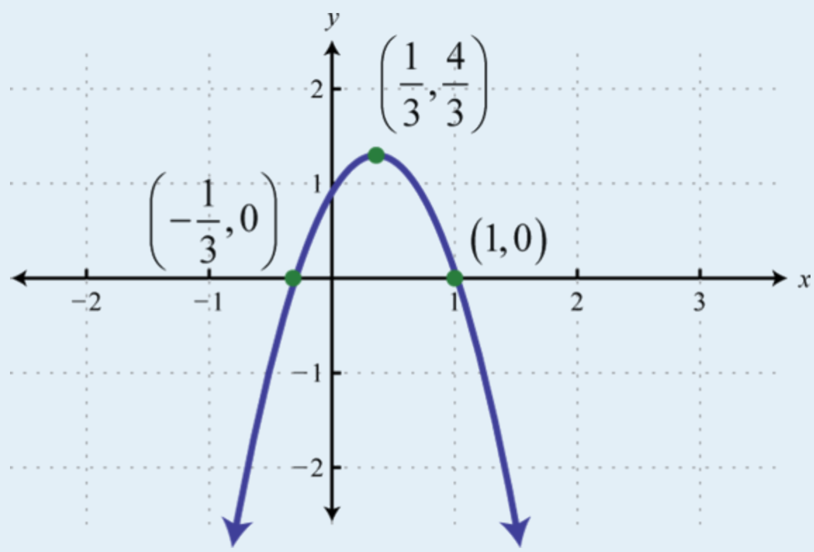
75. $x = (y - 2)^2 - 7$;



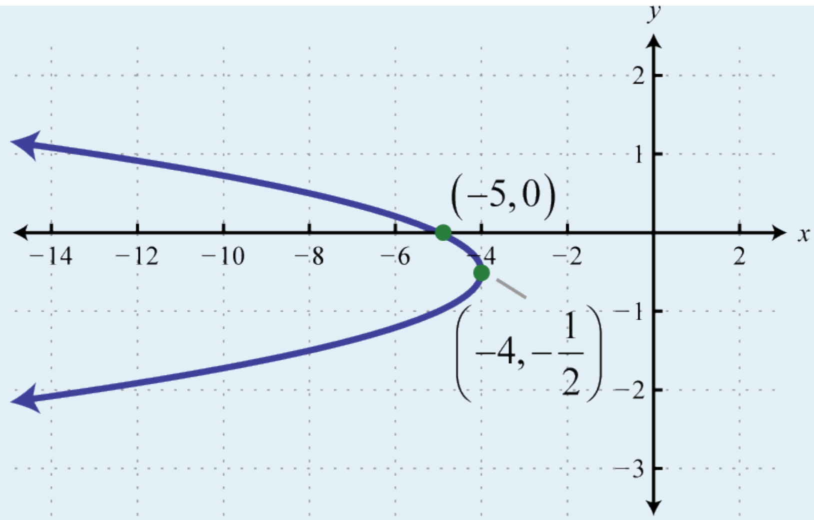
77. $x = -(y - 1)^2 + 6$;



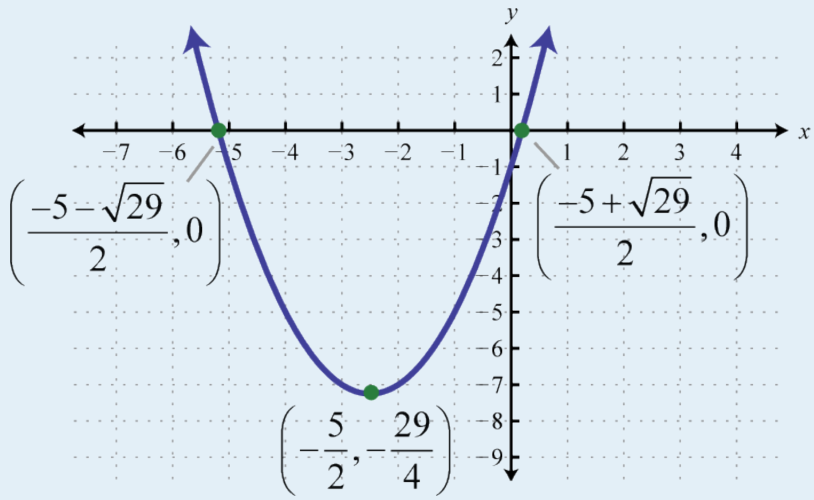
79. $y = -3\left(x - \frac{1}{3}\right)^2 + \frac{4}{3}$;



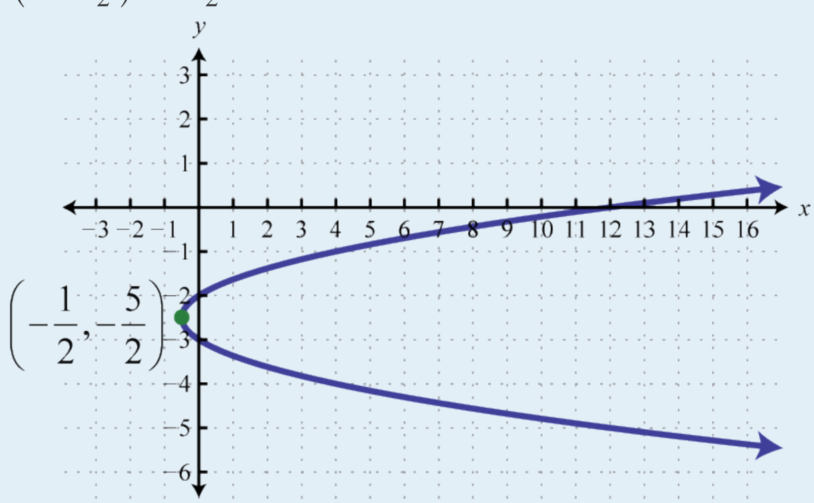
81. $x = -4\left(y + \frac{1}{2}\right)^2 - 4$;



83. $y = \left(x + \frac{5}{2}\right)^2 - \frac{29}{4}$;



85. $x = 2\left(y + \frac{5}{2}\right)^2 - \frac{1}{2}$;



87. Answer may vary

89. Answer may vary

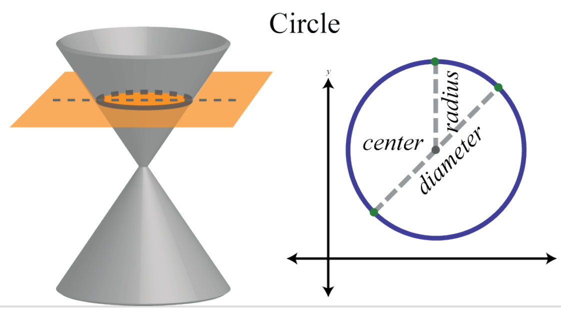
8.2 Circles

LEARNING OBJECTIVES

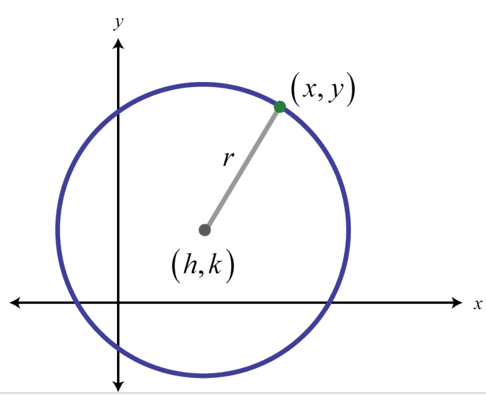
1. Graph a circle in standard form.
2. Determine the equation of a circle given its graph.
3. Rewrite the equation of a circle in standard form.

The Circle in Standard Form

A **circle**[8] is the set of points in a plane that lie a fixed distance, called the **radius**[9], from any point, called the center. The **diameter**[10] is the length of a line segment passing through the center whose endpoints are on the circle. In addition, a circle can be formed by the intersection of a cone and a plane that is perpendicular to the axis of the cone:



In a rectangular coordinate plane, where the center of a circle with radius r is (h, k), we have



8. A circle is the set of points in a plane that lie a fixed distance from a given point, called the center.
9. The fixed distance from the center of a circle to any point on the circle.
10. The length of a line segment passing through the center of a circle whose endpoints are on the circle.

Calculate the distance between (h, k) and (x, y) using the distance formula,

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

Squaring both sides leads us to the equation of a **circle in standard form**[11],

$$(x-h)^2 + (y-k)^2 = r^2$$

In this form, the center and radius are apparent. For example, given the equation $(x-2)^2 + (y+5)^2 = 16$ we have,

$$\begin{array}{ccccc} (x-h)^2 + & & (x-k)^2 & = & r^2 \\ \downarrow & & \downarrow & & \downarrow \\ (x-2)^2 + & [y- & (-5)]^2 & = & 4^2 \end{array}$$

In this case, the center is $(2, -5)$ and $r = 4$. More examples follow:

Equation	Center	Radius
$(x-3)^2 + (y-4)^2 = 25$	$(3, 4)$	$r = 5$

11. The equation of a circle written in the form $(x-h)^2 + (y-k)^2 = r^2$ where (h, k) is the center and r is the radius.

Equation	Center	Radius
$(x-1)^2+(y+2)^2=7$	$(1,-2)$	$r=\sqrt{7}$
$(x+4)^2+(y-3)^2=1$	$(-4,3)$	$r=1$
$x^2+(y+6)^2=8$	$(0,-6)$	$r=2\sqrt{2}$

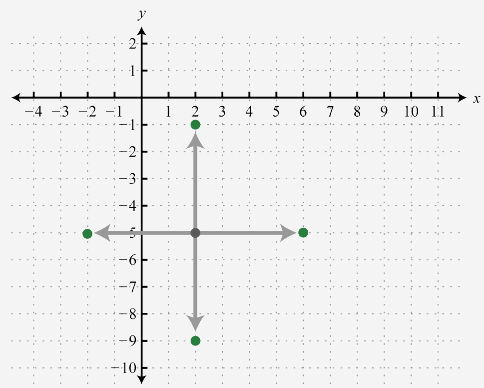
The graph of a circle is completely determined by its center and radius.

Example 1

Graph: $(x - 2)^2 + \left(y + 5\right)^2 = 16$.

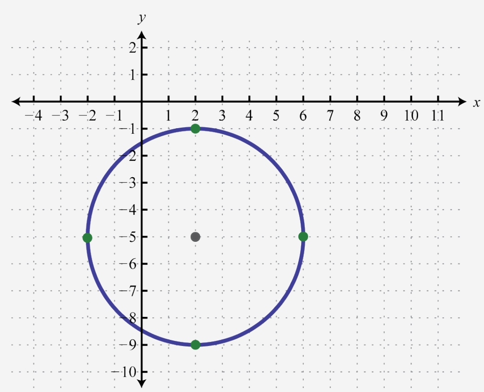
Solution:

Written in this form we can see that the center is $\left(2, -5\right)$ and that the radius $r = 4$units. From the center mark points 4 units up and down as well as 4 units left and right.



Then draw in the circle through these four points.

Answer:



As with any graph, we are interested in finding the *x*- and *y*-intercepts.

Example 2

Find the intercepts: $(x - 2)^2 + (y + 5)^2 = 16$.

Solution:

To find the y-intercepts set $x = 0$:

$$\begin{aligned}(x - 2)^2 + (y + 5)^2 &= 16\\ (0 - 2)^2 + (y + 5)^2 &= 16\\ 4 + (y + 5)^2 &= 16\end{aligned}$$

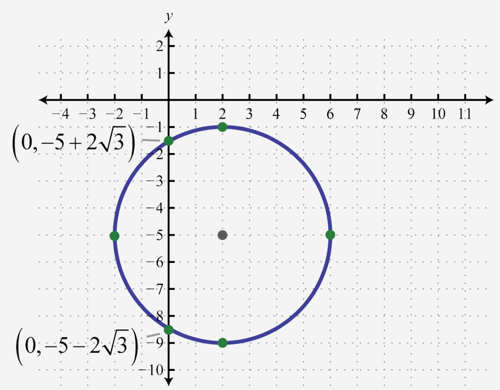
For this equation, we can solve by extracting square roots.

$$\begin{aligned}(y + 5)^2 &= 12\\ y + 5 &= \pm\sqrt{12}\\ y + 5 &= \pm 2\sqrt{3}\\ y &= -5 \pm 2\sqrt{3}\end{aligned}$$

Therefore, the y-intercepts are $\left(0, -5 - 2\sqrt{3}\right)$ and $\left(0, -5 + 2\sqrt{3}\right)$. To find the x-intercepts set $y = 0$:

$$
\begin{aligned}
(x-2)^2+(y+5)^2&=16\\
(x-2)^2+(0+5)^2&=16\\
(x-2)^2+25&=16\\
(x-2)^2&=-9\\
x-2&=\pm\sqrt{-9}\\
x&=2\pm 3i
\end{aligned}
$$

And because the solutions are complex we conclude that there are no real x-intercepts. Note that this does make sense given the graph.



Answer: x-intercepts: none; y-intercepts: $\left(0,-5-2\sqrt{3}\right)$ and $\left(0,-5+2\sqrt{3}\right)$

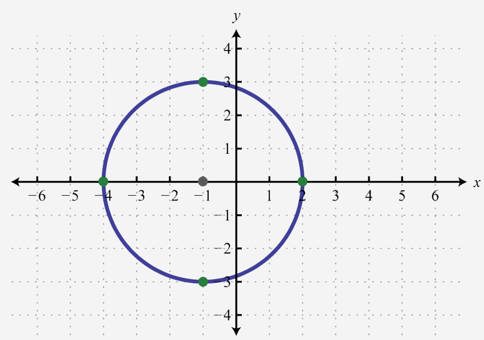
Given the center and radius of a circle, we can find its equation.

Example 3

Graph the circle with radius $r = 3$ units centered at $(-1, 0)$. Give its equation in standard form and determine the intercepts.

Solution:

Given that the center is $(-1, 0)$ and the radius is $r = 3$ we sketch the graph as follows:



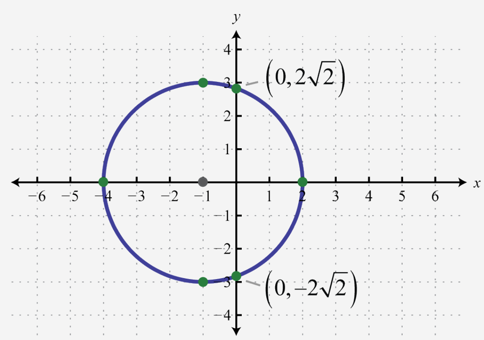
Substitute h, k, and r to find the equation in standard form. Since $(h, k) = (-1, 0)$ and $r = 3$ we have,

$$\begin{aligned}(x-h)^2 + (y-k)^2 &= r^2 \\ [x - (-1)]^2 + (y-0)^2 &= 3^2 \\ (x+1)^2 + y^2 &= 9\end{aligned}$$

The equation of the circle is $(x + 1)^2 + y^2 = 9$, use this to determine the y-intercepts.

$$
\begin{aligned}
(x+1)^2+y^2&=9 && \textit{Set } x=0 \textit{ to and solve for } y.\\
(0+1)^2+y^2&=9\\
1+y^2&=9\\
y^2&=8\\
y&=\pm\sqrt{8}\\
y&=\pm2\sqrt{2}
\end{aligned}
$$

Therefore, the *y*-intercepts are $\left(0, -2\sqrt{2}\right)$ and $\left(0, 2\sqrt{2}\right)$. To find the *x*-intercepts algebraically, set $y = 0$ and solve for *x*; this is left for the reader as an exercise.



Answer: Equation: $(x+1)^2 + y^2 = 9$; *y*-intercepts: $\left(0, -2\sqrt{2}\right)$ and $\left(0, 2\sqrt{2}\right)$; *x*-intercepts: $(-4, 0)$ and $(2, 0)$

Of particular importance is the **unit circle**[12],

$$x^2 + y^2 = 1$$

Or,

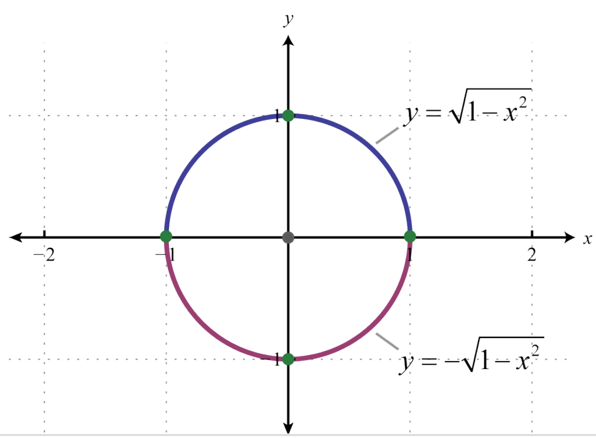
12. The circle centered at the origin with radius 1; its equation is $x^2 + y^2 = 1$.

$$(x-0)^2+(y-0)^2=1^2$$

In this form, it should be clear that the center is $(0, 0)$ and that the radius is 1 unit. Furthermore, if we solve for y we obtain two functions:

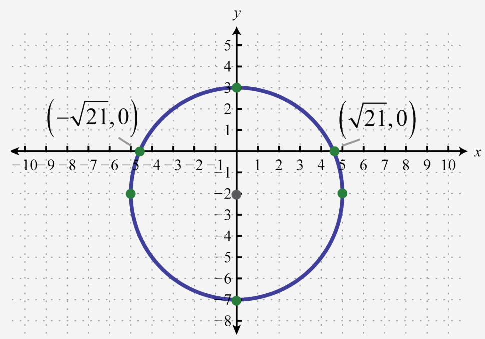
$$\begin{aligned} x^2+y^2&=1 \\ y^2&=1-x^2 \\ y&=\pm\sqrt{1-x^2} \end{aligned}$$

The function defined by $y=\sqrt{1-x^2}$ is the top half of the circle and the function defined by $y=-\sqrt{1-x^2}$ is the bottom half of the unit circle:



Try this! Graph and label the intercepts: $x^2 + (y+2)^2 = 25$.

Answer:



(click to see video)

The Circle in General Form

We have seen that the graph of a circle is completely determined by the center and radius which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of a **circle in general form**[13] follows:

$$x^2 + y^2 + cx + dy + e = 0$$

Here c, d, and e are real numbers. The steps for graphing a circle given its equation in general form follow.

13. The equation of a circle written in the form $x^2 + y^2 + cx + dy + e = 0$.

Example 4

Graph: $x^2 + y^2 + 6x - 8y + 13 = 0$.

Solution:

Begin by rewriting the equation in standard form.

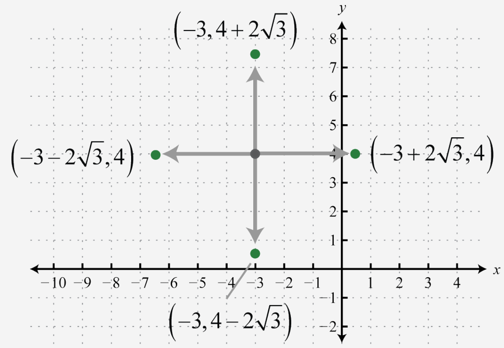
- **Step 1:** Group the terms with the same variables and move the constant to the right side. In this case, subtract 13 on both sides and group the terms involving x and the terms involving y as follows.

$$\begin{aligned} x^2 + y^2 + 6x - 8y + 13 &= 0 \\ \left(x^2 + 6x + \_\_\_\right) + \left(y^2 - 8y + \_\_\_\right) &= -13 \end{aligned}$$

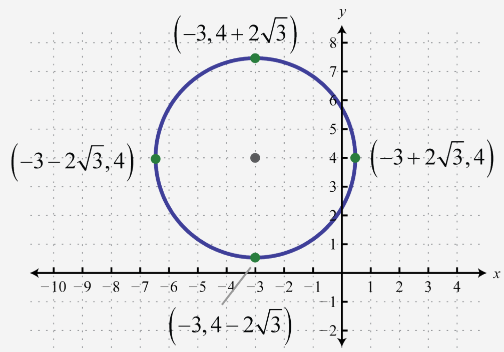
- **Step 2:** Complete the square for each grouping. The idea is to add the value that completes the square, $\left(\frac{b}{2}\right)^2$, to both sides for both groupings, and then factor. For the terms involving x use $\left(\frac{6}{2}\right)^2 = 3^2 = 9$ and for the terms involving y use $\left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$.

$$\begin{aligned} \left(x^2 + 6x + 9\right) + \left(y^2 - 8y + 16\right) &= -13 + 9 + 16 \\ (x+3)^2 + (y-4)^2 &= 12 \end{aligned}$$

- **Step 3:** Determine the center and radius from the equation in standard form. In this case, the center is $(-3, 4)$ and the radius $r = \sqrt{12} = 2\sqrt{3}$.
- **Step 4:** From the center, mark the radius vertically and horizontally and then sketch the circle through these points.


$\left(-3, 4+2\sqrt{3}\right)$
$\left(-3-2\sqrt{3}, 4\right)$
$\left(-3+2\sqrt{3}, 4\right)$
$\left(-3, 4-2\sqrt{3}\right)$
y
x

Answer:


$\left(-3, 4+2\sqrt{3}\right)$
$\left(-3-2\sqrt{3}, 4\right)$
$\left(-3+2\sqrt{3}, 4\right)$
$\left(-3, 4-2\sqrt{3}\right)$
y
x

Example 5

Determine the center and radius: $4x^2 + 4y^2 - 8x + 12y - 3 = 0$.

Solution:

We can obtain the general form by first dividing both sides by 4.

$$\frac{4x^2 + 4y^2 - 8x + 12y - 3}{4} = \frac{0}{4}$$
$$x^2 + y^2 - 2x + 3y - \frac{3}{4} = 0$$

Now that we have the general form for a circle, where both terms of degree two have a leading coefficient of 1, we can use the steps for rewriting it in standard form. Begin by adding $\frac{3}{4}$ to both sides and group variables that are the same.

$$\left(x^2 - 2x + \_\_\_\right) + \left(y^2 + 3y + \_\_\_\right) = \frac{3}{4}$$

Next complete the square for both groupings. Use $\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$ for the first grouping and $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ for the second grouping.

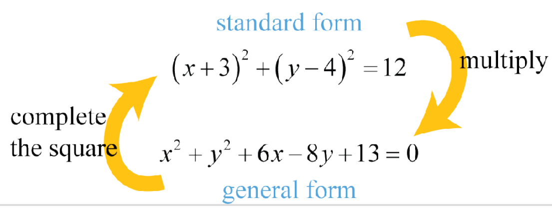
$$\left(x^2 - 2x + 1\right) + \left(y^2 + 3y + \frac{9}{4}\right) = \frac{3}{4} + 1 + \frac{9}{4}$$

$$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{16}{4}$$

$$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = 4$$

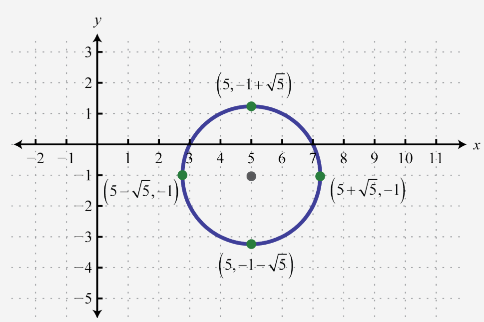
Answer: Center: $\left(1, -\frac{3}{2}\right)$; radius: $r = 2$

In summary, to convert from standard form to general form we multiply, and to convert from general form to standard form we complete the square.



Try this! Graph: $x^2 + y^2 - 10x + 2y + 21 = 0$.

Answer:



(click to see video)

KEY TAKEAWAYS

- The graph of a circle is completely determined by its center and radius.
- Standard form for the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$. The center is (h, k) and the radius measures *r* units.
- To graph a circle mark points *r* units up, down, left, and right from the center. Draw a circle through these four points.
- If the equation of a circle is given in general form $x^2 + y^2 + cx + dy + e = 0$, group the terms with the same variables, and complete the square for both groupings. This will result in standard form, from which we can read the circle's center and radius.
- We recognize the equation of a circle if it is quadratic in both *x* and *y* where the coefficient of the squared terms are the same.

TOPIC EXERCISES

PART A: THE CIRCLE IN STANDARD FORM

Determine the center and radius given the equation of a circle in standard form.

1. $(x-5)^2+(y+4)^2=64$
2. $(x+9)^2+(y-7)^2=121$
3. $x^2+(y+6)^2=4$
4. $(x-1)^2+y^2=1$
5. $(x+1)^2+(y+1)^2=7$
6. $(x+2)^2+(y-7)^2=8$

Determine the standard form for the equation of the circle given its center and radius.

7. Center $(5,7)$ with radius $r=7$.
8. Center $(-2,8)$ with radius $r=5$.
9. Center $(6,-11)$ with radius $r=\sqrt{2}$.
10. Center $(-4,-5)$ with radius $r=\sqrt{6}$.
11. Center $(0,-1)$ with radius $r=2\sqrt{5}$.
12. Center $(0,0)$ with radius $r=3\sqrt{10}$.

Graph.

13. $(x-1)^2+(y-2)^2=9$
14. $(x+3)^2+(y-3)^2=25$

15. $(x-2)^2+(y+6)^2=4$

16. $(x+6)^2+(y+4)^2=36$

17. $x^2+(y-4)^2=1$

18. $(x-3)^2+y^2=4$

19. $x^2+y^2=12$

20. $x^2+y^2=8$

21. $(x-7)^2+(y-6)^2=2$

22. $(x+2)^2+(y-5)^2=5$

23. $(x+3)^2+(y-1)^2=18$

24. $(x-3)^2+(y-2)^2=15$

Find the *x*- and *y*-intercepts.

25. $(x-1)^2+(y-2)^2=9$

26. $(x+5)^2+(y-3)^2=25$

27. $x^2+(y-4)^2=1$

28. $(x-3)^2+y^2=18$

29. $x^2+y^2=50$

30. $x^2+(y+9)^2=20$

31. $(x-4)^2+(y+5)^2=10$

32. $(x+10)^2+(y-20)^2=400$

Find the equation of the circle.

33. Circle with center $(1,-2)$ passing through $(3,-4)$.

34. Circle with center $(-4, -1)$ passing through $(0, -3)$.

35. Circle whose diameter is defined by $(5, 1)$ and $(-1, 7)$.

36. Circle whose diameter is defined by $(-5, 7)$ and $(-1, -5)$.

37. Circle with center $(5, -2)$ and area 9π square units.

38. Circle with center $(-8, -3)$ and circumference 12π square units.

39. Find the area of the circle with equation $(x + 12)^2 + (x - 5)^2 = 7$.

40. Find the circumference of the circle with equation $(x + 1)^2 + (y + 5)^2 = 8$.

PART B: THE CIRCLE IN GENERAL FORM

Rewrite in standard form and graph.

41. $x^2 + y^2 + 4x - 2y - 4 = 0$

42. $x^2 + y^2 - 10x + 2y + 10 = 0$

43. $x^2 + y^2 + 2x + 12y + 36 = 0$

44. $x^2 + y^2 - 14x - 8y + 40 = 0$

45. $x^2 + y^2 + 6y + 5 = 0$

46. $x^2 + y^2 - 12x + 20 = 0$

47. $x^2 + y^2 + 8x + 12y + 16 = 0$

48. $x^2 + y^2 - 20x - 18y + 172 = 0$

49. $4x^2 + 4y^2 - 4x + 8y + 1 = 0$

50. $9x^2 + 9y^2 + 18x + 6y + 1 = 0$

51. $x^2 + y^2 + 4x + 8y + 14 = 0$

52. $x^2 + y^2 - 2x - 4y - 15 = 0$

53. $x^2 + y^2 - x - 2y + 1 = 0$

54. $x^2 + y^2 - x + y - \frac{1}{2} = 0$

55. $4x^2 + 4y^2 + 8x - 12y + 5 = 0$

56. $9x^2 + 9y^2 + 12x - 36y + 4 = 0$

57. $2x^2 + 2y^2 + 6x + 10y + 9 = 0$

58. $9x^2 + 9y^2 - 6x + 12y + 4 = 0$

Given a circle in general form, determine the intercepts.

59. $x^2 + y^2 - 5x + 3y + 6 = 0$

60. $x^2 + y^2 + x - 2y - 7 = 0$

61. $x^2 + y^2 - 6y + 2 = 2$

62. $x^2 + y^2 - 6x - 8y + 5 = 0$

63. $2x^2 + 2y^2 - 3x - 9 = 0$

64. $3x^2 + 3y^2 + 8y - 16 = 0$

65. Determine the area of the circle whose equation is $x^2 + y^2 - 2x - 6y - 35 = 0.$

66. Determine the area of the circle whose equation is $4x^2 + 4y^2 - 12x - 8y - 59 = 0.$

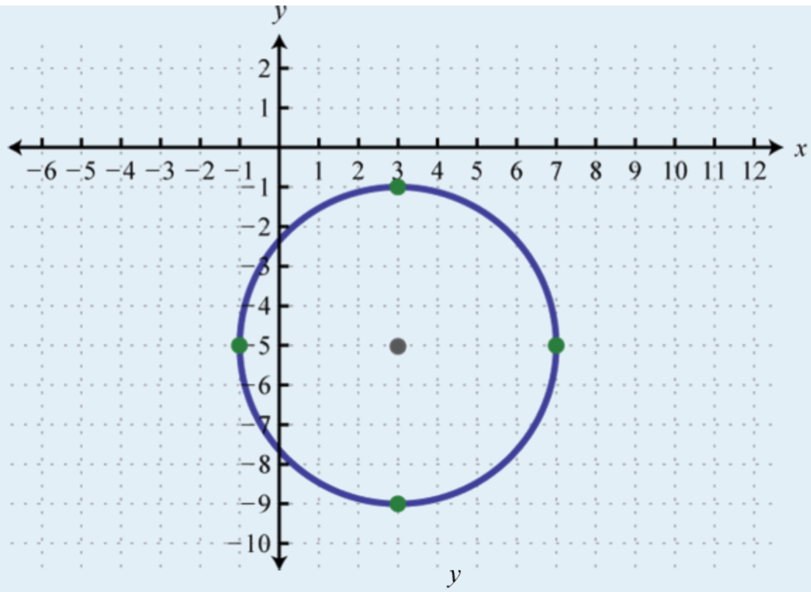
67. Determine the circumference of a circle whose equation is $x^2 + y^2 - 5x + 1 = 0.$

68. Determine the circumference of a circle whose equation is $x^2 + y^2 + 5x - 2y + 3 = 0.$

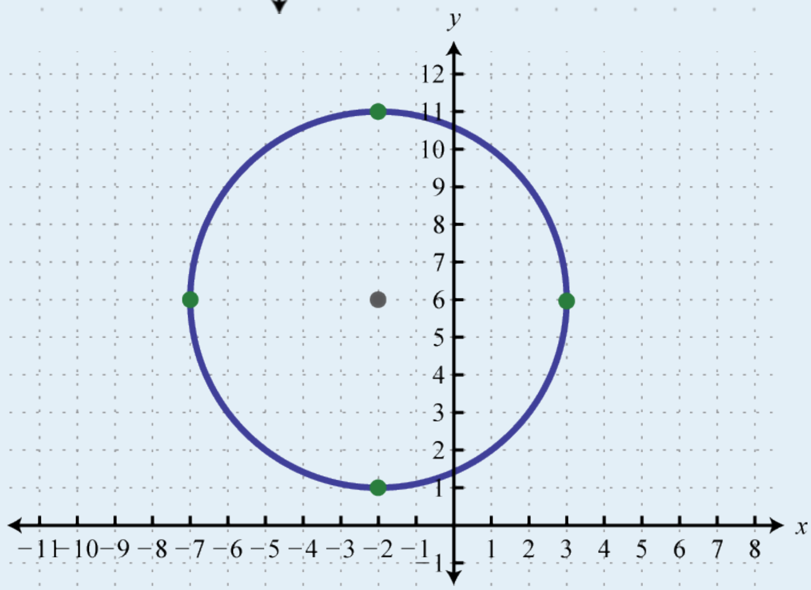
69. Find general form of the equation of a circle centered at $(-3, 5)$ passing through $(1, -2)$.

70. Find general form of the equation of a circle centered at $(-2, -3)$ passing through $(-1, 3)$.

Given the graph of a circle, determine its equation in general form.

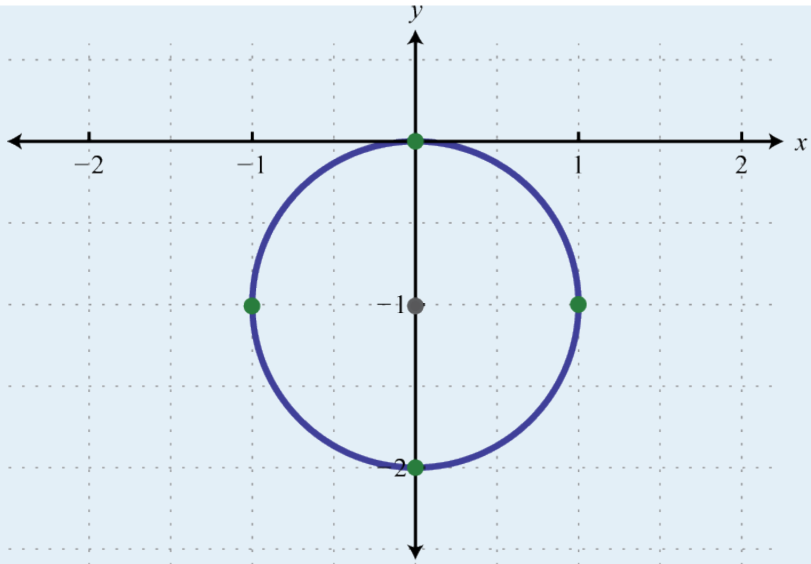

y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7 8 9 10 11 12
2 1 −1 −2 −3 −4 −5 −6 −7 −8 −9 −10

71.

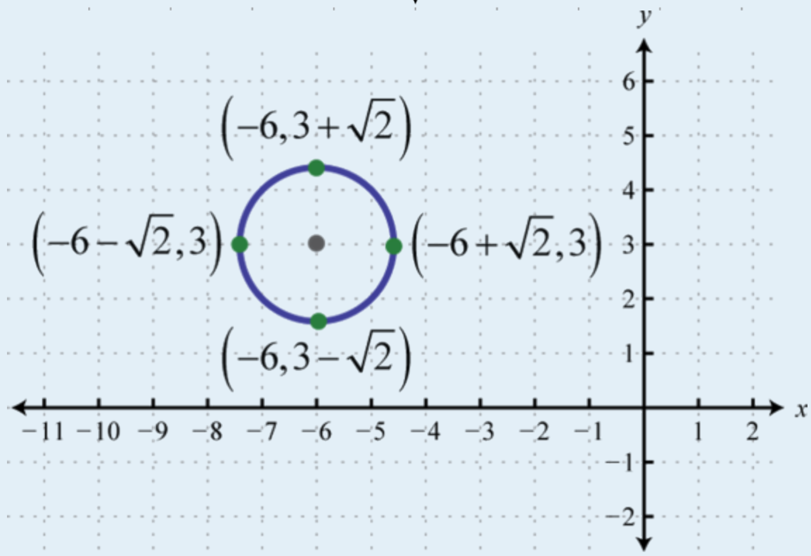

y
x
−11 −10 −9 −8 −7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7 8
12 11 10 9 8 7 6 5 4 3 2 1 −1

72.

73.



74.



PART C: DISCUSSION BOARD

75. Is the center of a circle part of the graph? Explain.

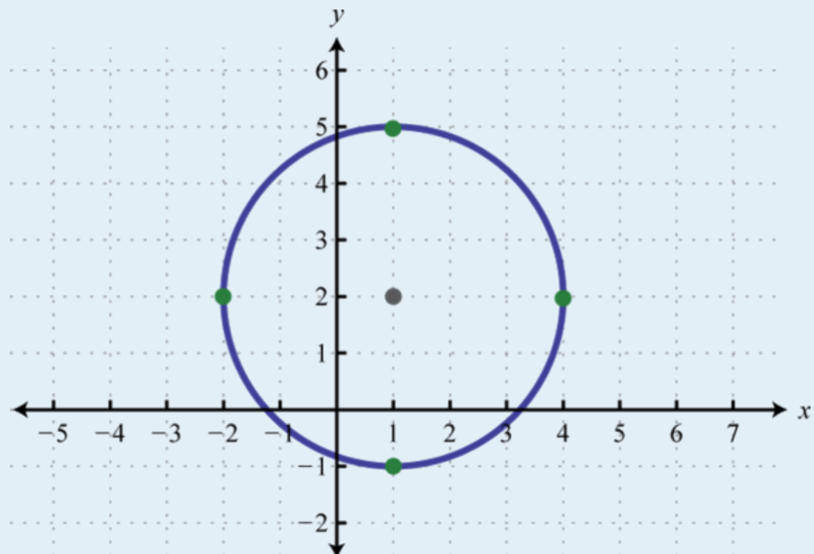
76. Make up your own circle, write it in general form, and graph it.

77. Explain how we can tell the difference between the equation of a parabola in general form and the equation of a circle in general form. Give an example.

78. Do all circles have intercepts? What are the possible numbers of intercepts? Illustrate your explanation with graphs.

ANSWERS

1. Center: $(5, -4)$; radius: $r = 8$
3. Center: $(0, -6)$; radius: $r = 2$
5. Center: $(-1, -1)$; radius: $r = \sqrt{7}$
7. $(x - 5)^2 + (y - 7)^2 = 49$
9. $(x - 6)^2 + (y + 11)^2 = 2$
11. $x^2 + (y + 1)^2 = 20$



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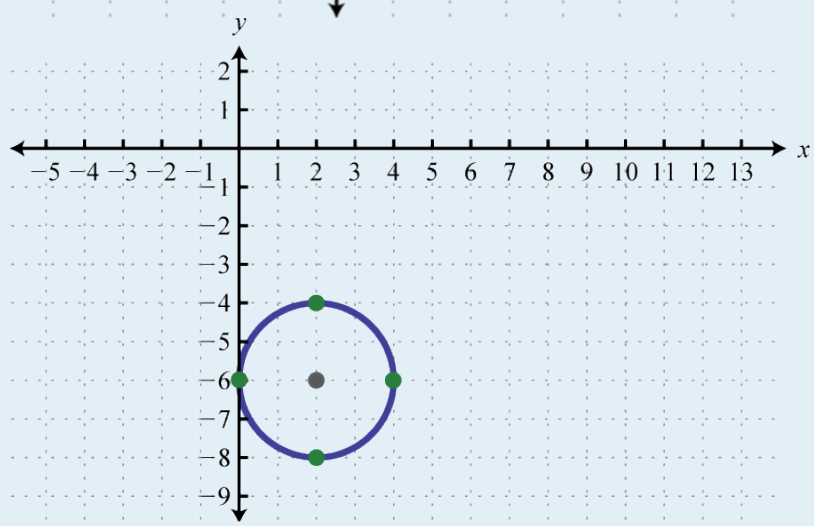
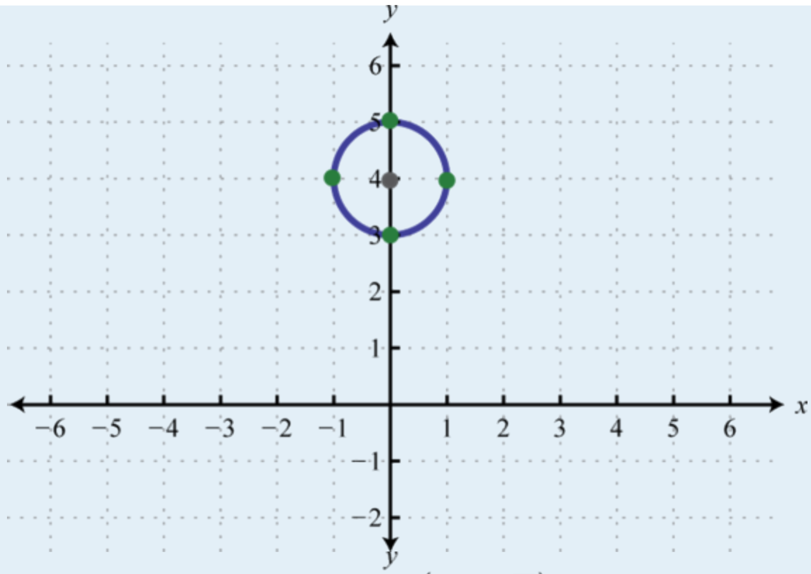
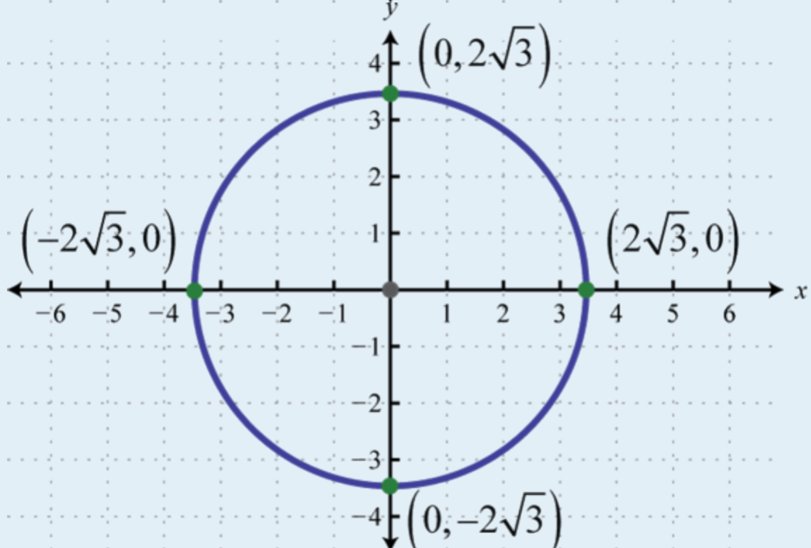
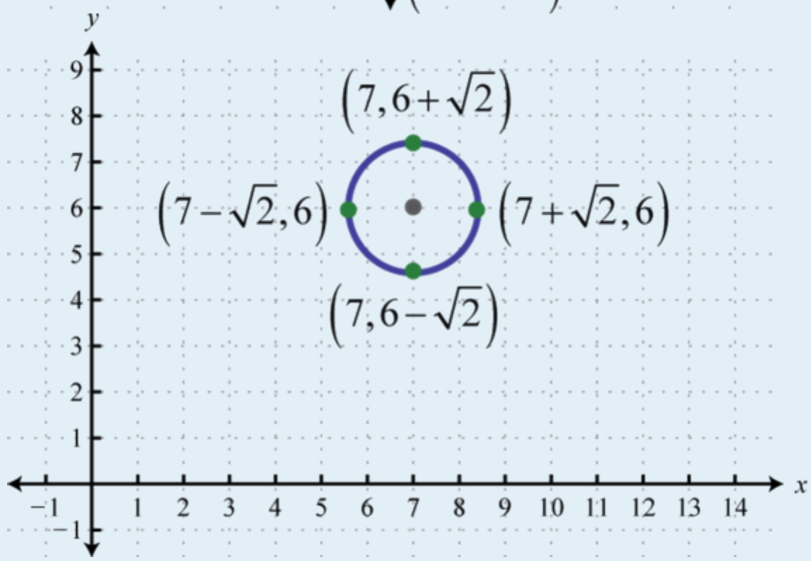
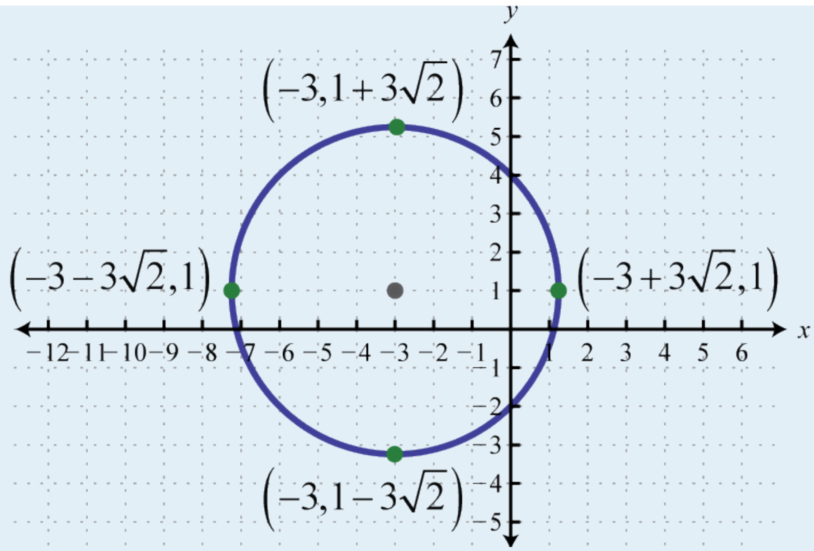


15.

17. 
y
x

19. 
y
x
$(0, 2\sqrt{3})$
$(-2\sqrt{3}, 0)$
$(2\sqrt{3}, 0)$
$(0, -2\sqrt{3})$

21. 
y
x
$(7, 6+\sqrt{2})$
$(7-\sqrt{2}, 6)$
$(7+\sqrt{2}, 6)$
$(7, 6-\sqrt{2})$

23. 

25. x-intercepts: $\left(1 \pm \sqrt{5}, 0\right)$; y-intercepts: $\left(0, 2 \pm 2\sqrt{2}\right)$

27. x-intercepts: none; y-intercepts: $(0, 3)$, $(0, 5)$

29. x-intercepts: $\left(\pm 5\sqrt{2}, 0\right)$; y-intercepts: $\left(0, \pm 5\sqrt{2}\right)$

31. x-intercepts: none; y-intercepts: none

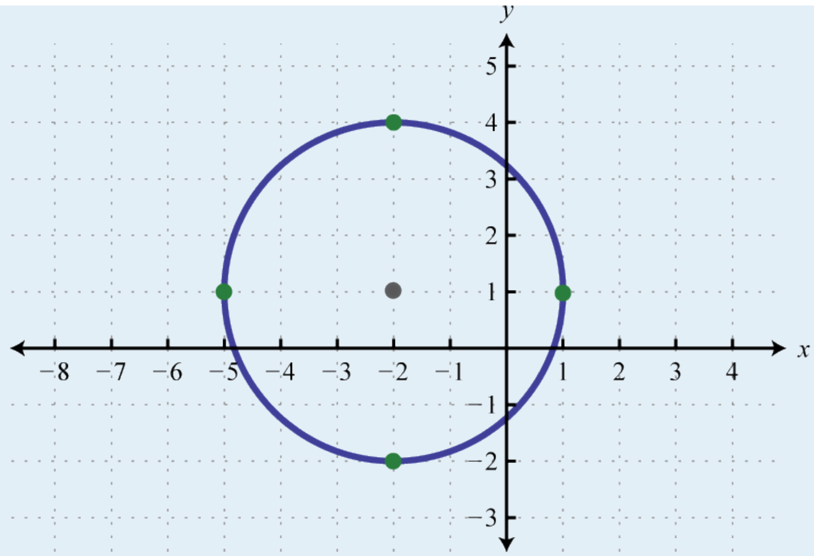
33. $(x-1)^2 + (y+2)^2 = 8$

35. $(x-2)^2 + (y-4)^2 = 18$

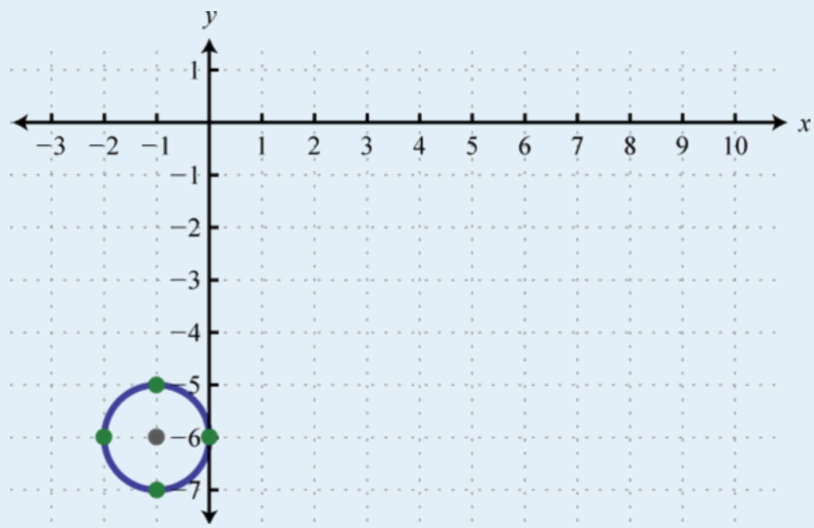
37. $(x-5)^2 + (y+2)^2 = 9$

39. 7π square units

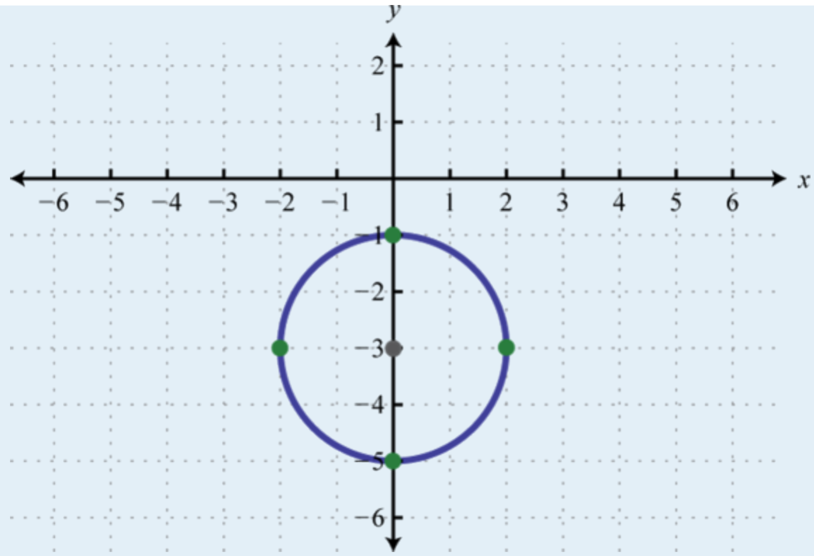
41. $(x+2)^2 + (y-1)^2 = 9$;



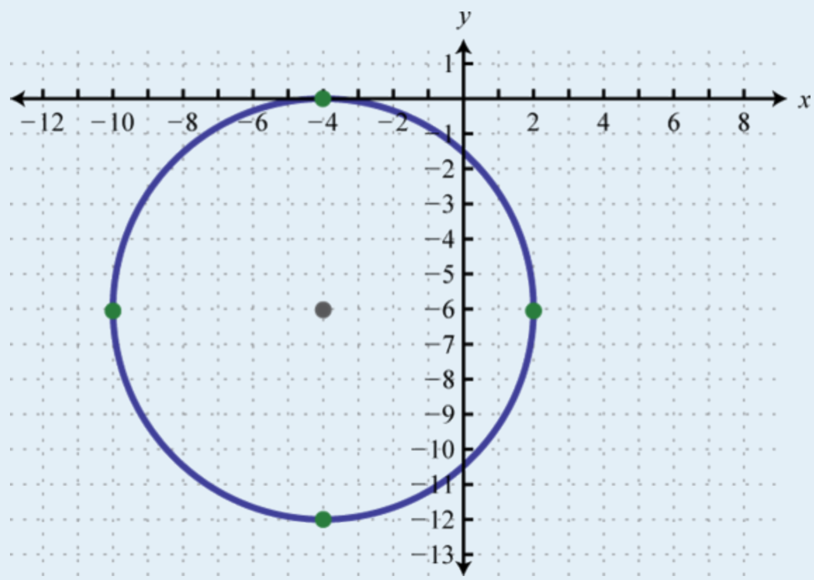
43. $(x+1)^2+(y+6)^2=1$;



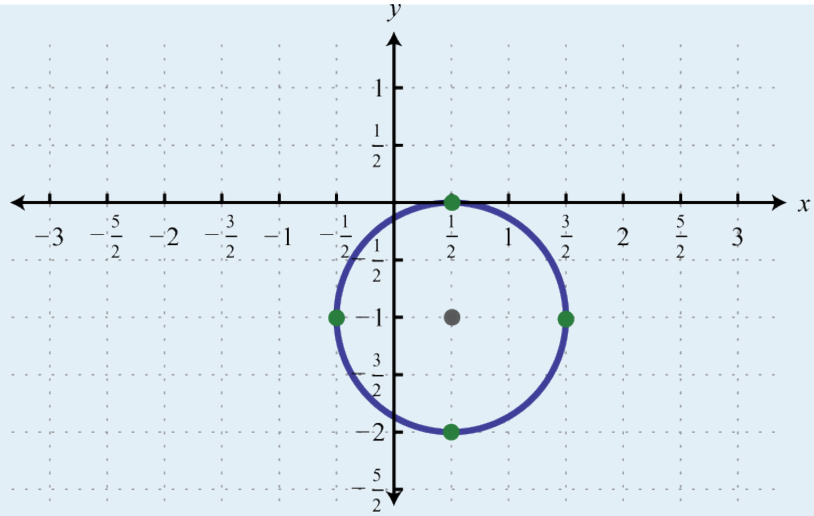
45. $x^2+(y+3)^2=4$;



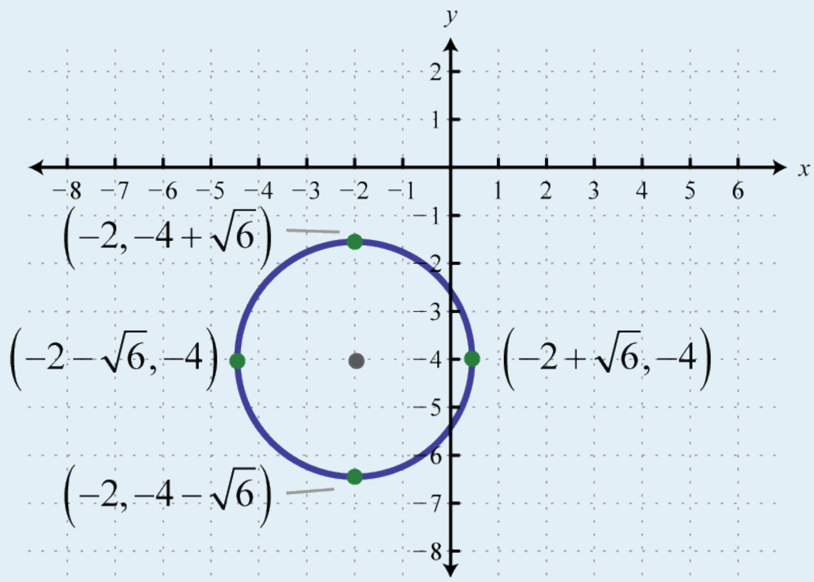
47. $(x+4)^2 + \left(y+6\right)^2 = 36;$



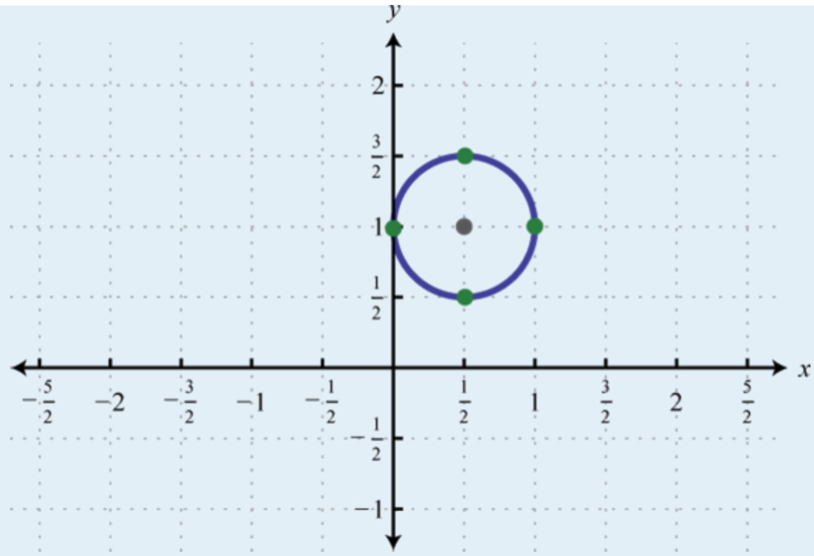
49. $\left(x - \frac{1}{2}\right)^2 + \left(y+1\right)^2 = 1;$



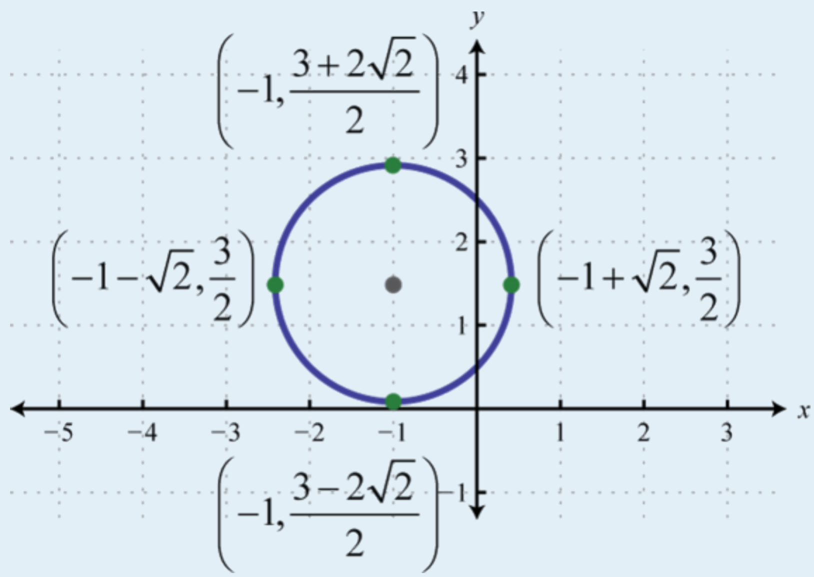
51. $(x+2)^2+(y-4)^2=6;$



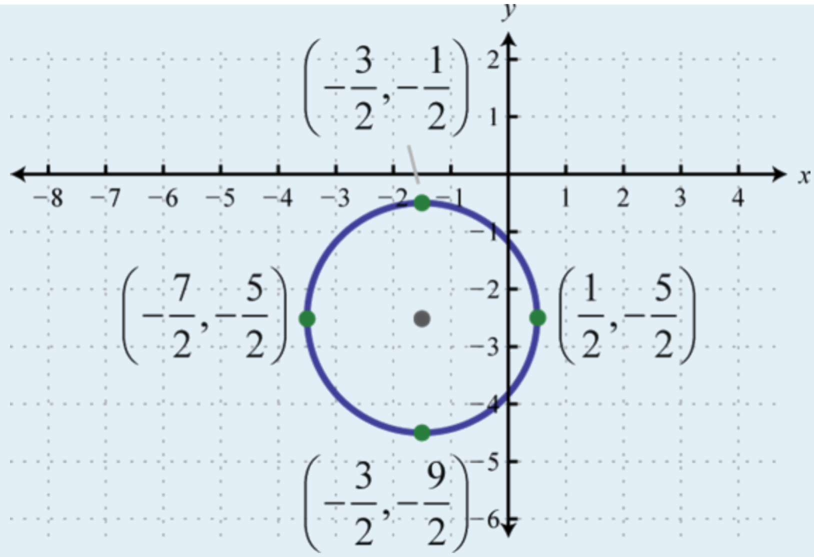
53. $\left(x-\frac{1}{2}\right)^2+(y-1)^2=\frac{1}{4};$



55. $(x+1)^2+\left(y-\frac{3}{2}\right)^2=2$;



57. $\left(x+\frac{3}{2}\right)^2+\left(y+\frac{5}{2}\right)^2=4$;



59. x-intercepts: $(2, 0), (3, 0)$; y-intercepts: none

61. x-intercepts: $(0, 0)$; y-intercepts: $(0, 0), (0, 6)$

63. x-intercepts: $\left(-\frac{3}{2}, 0\right), (3, 0)$; y-intercepts: $\left(0, \pm\frac{3\sqrt{2}}{2}\right)$

65. 45π square units

67. $\pi\sqrt{21}$ units

69. $x^2 + y^2 + 6x - 10y - 31 = 0$

71. $x^2 + y^2 - 6x + 10y + 18 = 0$

73. $x^2 + y^2 + 2y = 0$

75. Answer may vary

77. Answer may vary

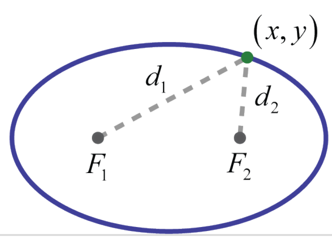
8.3 Ellipses

LEARNING OBJECTIVES

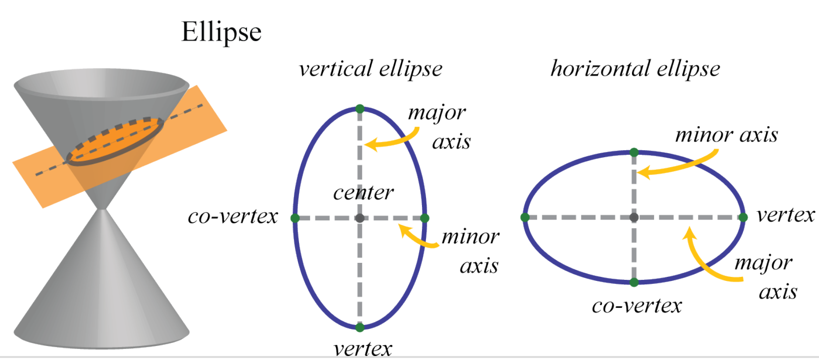
1. Graph an ellipse in standard form.
2. Determine the equation of an ellipse given its graph.
3. Rewrite the equation of an ellipse in standard form.

The Ellipse in Standard Form

An **ellipse**[14] is the set of points in a plane whose distances from two fixed points, called foci, have a sum that is equal to a positive constant. In other words, if points F_1 and F_2 are the foci (plural of focus) and d is some given positive constant then (x, y) is a point on the ellipse if $d = d_1 + d_2$ as pictured below:



In addition, an ellipse can be formed by the intersection of a cone with an oblique plane that is not parallel to the side of the cone and does not intersect the base of the cone. Points on this oval shape where the distance between them is at a maximum are called **vertices**[15] and define the **major axis**[16]. The center of an ellipse is the midpoint between the vertices. The **minor axis**[17] is the line segment through the center of an ellipse defined by two points on the ellipse where the distance between them is at a minimum. The endpoints of the minor axis are called **co-vertices**[18].



14. The set of points in a plane whose distances from two fixed points have a sum that is equal to a positive constant.

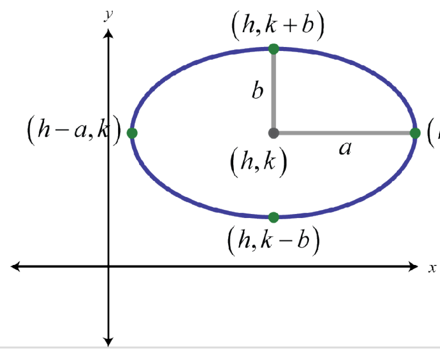
15. Points on the ellipse that mark the endpoints of the major axis.

16. The line segment through the center of an ellipse defined by two points on the ellipse where the distance between them is a maximum.

17. The line segment through the center of an ellipse defined by two points on the ellipse where the distance between them is a minimum.

18. Points on the ellipse that mark the endpoints of the minor axis.

If the major axis of an ellipse is parallel to the *x*-axis in a rectangular coordinate plane, we say that the ellipse is horizontal. If the major axis is parallel to the *y*-axis, we say that the ellipse is vertical. In this section, we are only concerned with sketching these two types of ellipses. However, the ellipse has many real-world applications and further research on this rich subject is encouraged. In a rectangular coordinate plane, where the center of a horizontal ellipse is (h, k), we have



As pictured $a > b$ where *a*, one-half of the length of the major axis, is called the **major radius**[19]. And *b*, one-half of the length of the minor axis, is called the **minor radius**[20]. The equation of an **ellipse in standard form**[21] follows:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The vertices are $(h \pm a, k)$ and $(h, k \pm b)$ and the orientation depends on *a* and *b*. If $a > b$, then the ellipse is horizontal as shown above and if $a < b$, then the ellipse is vertical and *b* becomes the major radius. What do you think happens when $a = b$?

19. One-half of the length of the major axis.

20. One-half of the length of the minor axis.

21. The equation of an ellipse written in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.The center is (h, k) and the larger of *a* and *b* is the major radius and the smaller is the minor radius.

Equation	Center	a	b	Orientation
$\frac{(x-1)^2}{4} + \frac{(y-8)^2}{9} = 1$	$(1, 8)$	$a = 2$	$b = 3$	Vertical
$\frac{(x-3)^2}{2} + \frac{(y+5)^2}{16} = 1$	$(3, -5)$	$a = \sqrt{2}$	$b = 4$	Vertical
$\frac{(x+1)^2}{1} + \frac{(y-7)^2}{8} = 1$	$(-1, 7)$	$a = 1$	$b = 2\sqrt{2}$	Vertical
$\frac{x^2}{25} + \frac{(y+6)^2}{10} = 1$	$(0, -6)$	$a = 5$	$b = \sqrt{10}$	Horizontal

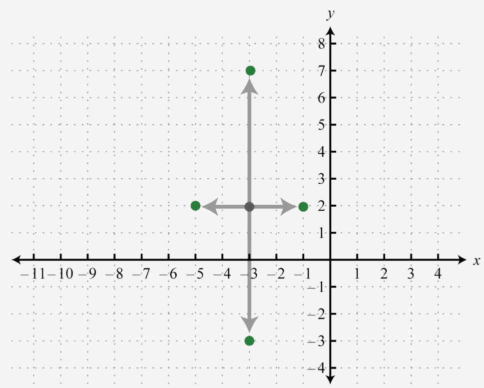
The graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius, all of which can be determined from its equation written in standard from.

Example 1

Graph: $\frac{(x+3)^2}{4} + \frac{(y-2)^2}{25} = 1$.

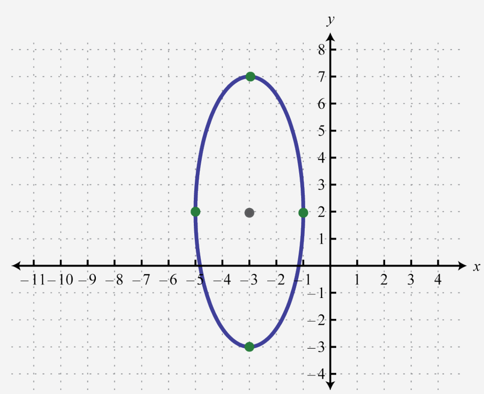
Solution:

Written in this form we can see that the center of the ellipse is $(-3, 2)$, $a = \sqrt{4} = 2$, and $b = \sqrt{25} = 5$. From the center mark points 2 units to the left and right and 5 units up and down.



Then draw an ellipse through these four points.

Answer:



As with any graph, we are interested in finding the *x*- and *y*-intercepts.

Example 2

Find the intercepts: $\frac{(x+3)^2}{4} + \frac{(y-2)^2}{25} = 1$.

Solution:

To find the x-intercepts set $y = 0$:

$$\begin{aligned}\frac{(x+3)^2}{4} + \frac{(0-2)^2}{25} &= 1\\ \frac{(x+3)^2}{4} + \frac{4}{25} &= 1\\ \frac{(x+3)^2}{4} &= 1 - \frac{4}{25}\\ \frac{(x+3)^2}{4} &= \frac{21}{25}\end{aligned}$$

At this point we extract the root by applying the square root property.

$$\begin{aligned}\frac{x+3}{2} &= \pm\sqrt{\frac{21}{25}}\\ x+3 &= \pm\frac{2\sqrt{21}}{5}\\ x &= -3 \pm \frac{2\sqrt{21}}{5} = \frac{-15 \pm 2\sqrt{21}}{5}\end{aligned}$$

Setting $x = 0$ and solving for y leads to complex solutions, therefore, there are no y-intercepts. This is left as an exercise.

Answer: x-intercepts: $\left(\frac{-15 \pm 2\sqrt{21}}{5}, 0\right)$; y-intercepts: none.

Unlike a circle, standard form for an ellipse requires a 1 on one side of its equation.

Example 3

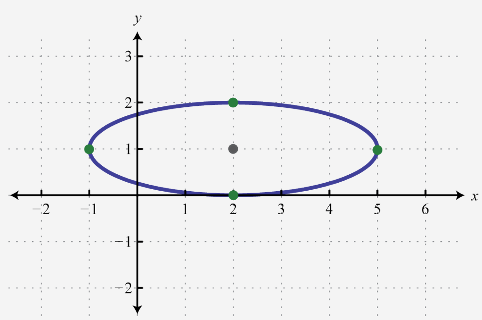
Graph and label the intercepts: $(x-2)^2+9(y-1)^2=9$.

Solution:

To obtain standard form, with 1 on the right side, divide both sides by 9.

$$\frac{(x-2)^2+9(y-1)^2}{9}=\frac{9}{9}$$
$$\frac{(x-2)^2}{9}+\frac{9(y-1)^2}{9}=\frac{9}{9}$$
$$\frac{(x-2)^2}{9}+\frac{(y-1)^2}{1}=1$$

Therefore, the center of the ellipse is $(2, 1)$, $a=\sqrt{9}=3$, and $b=\sqrt{1}=1$. The graph follows:

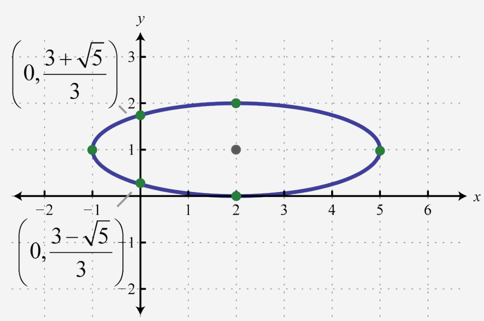


To find the intercepts we can use the standard form $\frac{(x-2)^2}{9}+(y-1)^2=1$:

x-intercepts set $y=0$	y-intercepts set $x=0$
$\frac{(x-2)^2}{9}+(0-1)^2=1$ $\frac{(x-2)^2}{9}+1=1$ $(x-2)^2=0$ $x-2=0$ $x=2$	$\frac{(0-2)^2}{9}+(y-1)^2=1$ $\frac{4}{9}+(y-1)^2=1$ $(y-1)^2=\frac{5}{9}$ $y-1=\pm\sqrt{\frac{5}{9}}$ $y=1\pm\frac{\sqrt{5}}{3}=\frac{3\pm\sqrt{5}}{3}$

Therefore the x-intercept is $(2,0)$ and the y-intercepts are $\left(0,\frac{3+\sqrt{5}}{3}\right)$ and $\left(0,\frac{3-\sqrt{5}}{3}\right)$.

Answer:



Consider the ellipse centered at the origin,

$$x^2 + \frac{y^2}{4} = 1$$

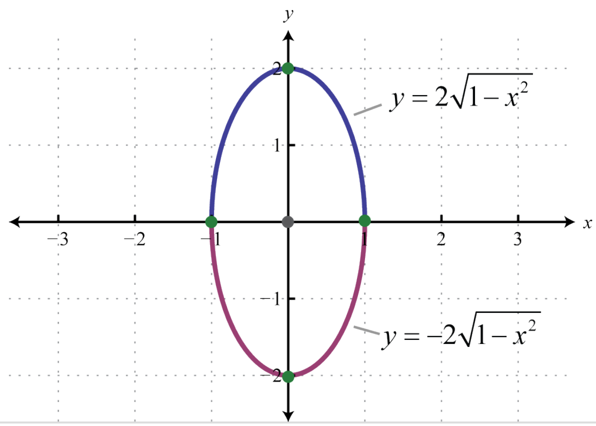
Given this equation we can write,

$$\frac{(x-0)^2}{1^2} + \frac{(y-0)^2}{2^2} = 1$$

In this form, it is clear that the center is $(0, 0)$, $a = 1$, and $b = 2$. Furthermore, if we solve for y we obtain two functions:

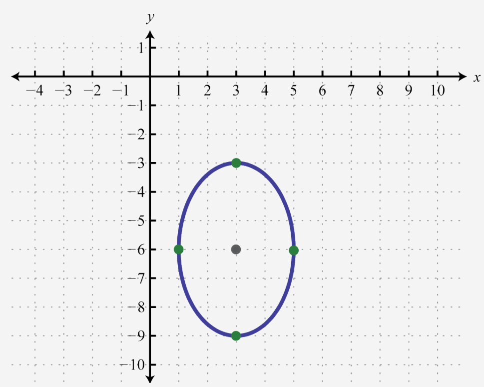
$$\begin{aligned} x^2 + \frac{y^2}{4} &= 1 \\ \frac{y^2}{4} &= 1 - x^2 \\ y^2 &= 4\left(1 - x^2\right) \\ y &= \pm\sqrt{4\left(1 - x^2\right)} \\ y &= \pm 2\sqrt{1 - x^2} \end{aligned}$$

The function defined by $y = 2\sqrt{1 - x^2}$ is the top half of the ellipse and the function defined by $y = -2\sqrt{1 - x^2}$ is the bottom half.



Try this! Graph: $9(x-3)^2 + 4(y+2)^2 = 36$.

Answer:



(click to see video)

The Ellipse in General Form

We have seen that the graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius; which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of an **ellipse in general form**[22] follows,

$$px^2 + qy^2 + cx + dy + e = 0$$

22. The equation of an ellipse written in the form $px^2 + qy^2 + cx + dy + e = 0$ where $p, q > 0$.

where $p, q > 0$. The steps for graphing an ellipse given its equation in general form are outlined in the following example.

Example 4

Graph: $2x^2 + 9y^2 + 16x - 90y + 239 = 0$.

Solution:

Begin by rewriting the equation in standard form.

- **Step 1:** Group the terms with the same variables and move the constant to the right side. Factor so that the leading coefficient of each grouping is 1.

$$\begin{aligned} 2x^2 + 9y^2 + 16x - 90y + 239 &= 0 \\ \left(2x^2 + 16x + \_\_\_\right) + \left(9y^2 - 90y + \_\_\_\right) &= -239 \\ 2\left(x^2 + 8x + \_\_\_\right) + 9\left(y^2 - 10y + \_\_\_\right) &= -239 \end{aligned}$$

- **Step 2:** Complete the square for each grouping. In this case, for the terms involving x use $\left(\frac{8}{2}\right)^2 = 4^2 = 16$ and for the terms involving y use $\left(\frac{-10}{2}\right)^2 = (-5)^2 = 25$. The factor in front of the grouping affects the value used to balance the equation on the right side:

$$2\left(x^2 + 8x \color{blue}{+ 16}\right) + 9\left(y^2 - 10y \color{green}{+ 25}\right) = -239 \color{blue}{+ 32} \color{green}{+ 225}$$

Because of the distributive property, adding 16 inside of the first grouping is equivalent to adding $2 \cdot 16 = 32$. Similarly, adding 25 inside of the second grouping is equivalent to adding $9 \cdot 25 = 225$. Now factor and then divide to obtain 1 on the right side.

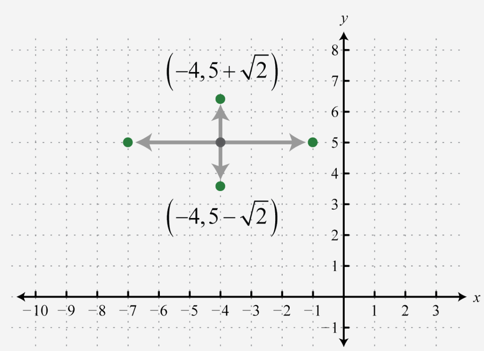
$$2(x+4)^2 + 9(y-5)^2 = 18$$

$$\frac{2(x+4)^2 + 9(y-5)^2}{18} = \frac{18}{18}$$

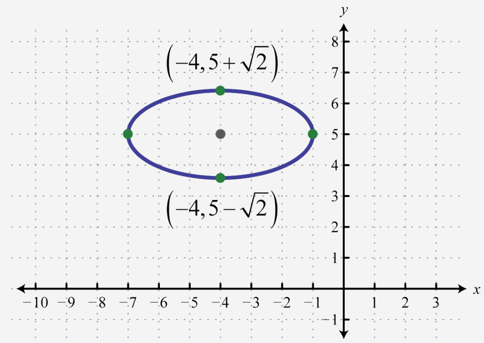
$$\frac{2(x+4)^2}{18} + \frac{9(y-5)^2}{18} = \frac{18}{18}$$

$$\frac{(x+4)^2}{9} + \frac{(y-5)^2}{2} = 1$$

- **Step 3:** Determine the center, a, and b. In this case, the center is $(-4, 5)$, $a = \sqrt{9} = 3$, and $b = \sqrt{2}$.
- **Step 4:** Use a to mark the vertices left and right of the center, use b to mark the vertices up and down from the center, and then sketch the graph. In this case, the vertices along the minor axes $\left(-4, 5 \pm \sqrt{2}\right)$ are not apparent and should be labeled.



Answer:



$\left(-4, 5+\sqrt{2}\right)$
$\left(-4, 5-\sqrt{2}\right)$
y
x

Example 5

Determine the center of the ellipse as well as the lengths of the major and minor axes: $5x^2 + y^2 - 3x + 40 = 0$.

Solution:

In this example, we only need to complete the square for the terms involving x.

$$\begin{aligned} 5x^2 + y^2 - 30x + 40 &= 0 \\ \left(5x^2 - 30x + \underline{\quad}\right) + y^2 &= -40 \\ 5\left(x^2 - 6x + \underline{\quad}\right) + y^2 &= -40 \end{aligned}$$

Use $\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$ for the first grouping to be balanced by $5 \cdot 9 = 45$ on the right side.

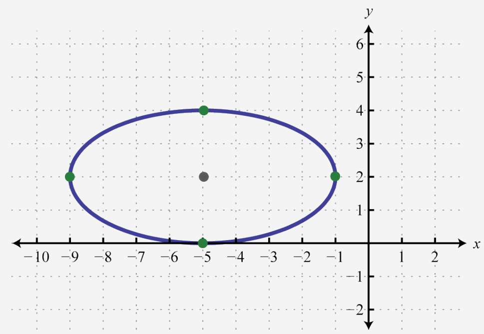
$$\begin{aligned} 5\left(x^2 - 6x + 9\right) + y^2 &= -40 + 45 \\ 5(x-3)^2 + y^2 &= 5 \\ \frac{5(x-3)^2 + y^2}{5} &= \frac{5}{5} \\ \frac{(x-3)^2}{1} + \frac{y^2}{5} &= 1 \end{aligned}$$

Here, the center is $(3, 0)$, $a = \sqrt{1} = 1$, and $b = \sqrt{5}$. Because b is larger than a, the length of the major axis is $2b$ and the length of the minor axis is $2a$.

Answer: Center: $(3, 0)$; major axis: $2\sqrt{5}$ units; minor axis: 2 units.

Try this! Graph: $x^2 + 4y^2 + 10x - 16y + 25 = 0.$

Answer:



(click to see video)

KEY TAKEAWAYS

- The graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius.
- The center, orientation, major radius, and minor radius are apparent if the equation of an ellipse is given in standard form: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$
- To graph an ellipse, mark points *a* units left and right from the center and points *b* units up and down from the center. Draw an ellipse through these points.
- The orientation of an ellipse is determined by *a* and *b*. If $a > b$ then the ellipse is wider than it is tall and is considered to be a horizontal ellipse. If $a < b$ then the ellipse is taller than it is wide and is considered to be a vertical ellipse.
- If the equation of an ellipse is given in general form $px^2 + qy^2 + cx + dy + e = 0$ where $p, q > 0$, group the terms with the same variables, and complete the square for both groupings.
- We recognize the equation of an ellipse if it is quadratic in both *x* and *y* and the coefficients of each square term have the same sign.

TOPIC EXERCISES

PART A: THE ELLIPSE IN STANDARD FORM

Given the equation of an ellipse in standard form, determine its center, orientation, major radius, and minor radius.

1. $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{49} = 1$
2. $\frac{(x+3)^2}{64} + \frac{(y-2)^2}{9} = 1$
3. $\frac{x^2}{3} + (y+9)^2 = 1$
4. $\frac{(x-1)^2}{8} + y^2 = 1$
5. $4(x+5)^2 + 9(y+5)^2 = 36$
6. $16(x-1)^2 + 3(y+10)^2 = 48$

Determine the standard form for the equation of an ellipse given the following information.

7. Center $(3, 4)$ with $a = 5$ and $b = 2$.
8. Center $(-1, 9)$ with $a = 7$ and $b = 3$.
9. Center $(5, -1)$ with $a = \sqrt{6}$ and $b = 2\sqrt{3}$.
10. Center $(-7, -2)$ with $a = 5\sqrt{2}$ and $b = \sqrt{7}$.
11. Center $(0, -3)$ with $a = 1$ and $b = \sqrt{5}$.
12. Center $(0, 0)$ with $a = \sqrt{2}$ and $b = 4$.

Graph.

13. $\frac{(x-4)^2}{4} + \frac{(y+2)^2}{9} = 1$
14. $\frac{(x+1)^2}{25} + \frac{(y-2)^2}{4} = 1$

15. $\frac{(x-5)^2}{16} + \frac{(y+6)^2}{1} = 1$

16. $\frac{(x+4)^2}{4} + \frac{(y+3)^2}{36} = 1$

17. $\frac{(x-2)^2}{9} + \frac{(y-1)^2}{64} = 1$

18. $\frac{(x+1)^2}{49} + (y+3)^2 = 1$

19. $4(x+3)^2 + 9(y-3)^2 = 36$

20. $16x^2 + (y-1)^2 = 16$

21. $4(x-2)^2 + 25y^2 = 100$

22. $81x^2 + y^2 = 81$

23. $\frac{(x-2)^2}{8} + \frac{(y-4)^2}{9} = 1$

24. $\frac{(x+1)^2}{4} + \frac{(y-1)^2}{12} = 1$

25. $\frac{(x-6)^2}{2} + \frac{(y+2)^2}{5} = 1$

26. $\frac{(x+3)^2}{18} + \frac{(y-5)^2}{3} = 1$

27. $3x^2 + 2(y-3)^2 = 6$

28. $5(x+1)^2 + 3y^2 = 15$

29. $4x^2 + 6y^2 = 24$

30. $5x^2 + 10y^2 = 50$

Find the *x*- and *y*-intercepts.

31. $\frac{(x-3)^2}{4} + \frac{(y-2)^2}{9} = 1$

32. $\frac{(x+3)^2}{16} + \frac{(y-7)^2}{9} = 1$

33. $\frac{(x-2)^2}{4} + \frac{(y+6)^2}{36} = 1$

34. $\frac{(x+1)^2}{25} + \frac{(y-1)^2}{9} = 1$

35. $5x^2 + 2(y-4)^2 = 20$

36. $4(x-3)^2 + 9y^2 = 72$

37. $5x^2 + 2y^2 = 10$

38. $3x^2 + 4y^2 = 24$

Find the equation of the ellipse.

39. Ellipse with vertices $(\pm 5, 0)$ and $(0, \pm 6)$.

40. Ellipse whose major axis has vertices $(2, 9)$ and $(2, -1)$ and minor axis has vertices $(-2, 4)$ and $(6, 4)$.

41. Ellipse whose major axis has vertices $(-8, -2)$ and $(0, -2)$ and minor axis has a length of 4 units.

42. Ellipse whose major axis has vertices $(-2, 2)$ and $(-2, 8)$ and minor axis has a length of 2 units.

PART B: THE ELLIPSE IN GENERAL FORM

Rewrite in standard form and graph.

43. $4x^2 + 9y^2 + 8x - 36y + 4 = 0$

44. $9x^2 + 25y^2 - 18x + 100y - 116 = 0$

45. $4x^2 + 49y^2 + 24x + 98y - 111 = 0$

46. $9x^2 + 4y^2 - 72x + 24y + 144 = 0$

47. $x^2 + 64y^2 - 12x + 128y + 36 = 0$

48. $16x^2 + y^2 - 96x - 4y + 132 = 0$

49. $36x^2 + 4y^2 - 40y - 44 = 0$

50. $x^2 + 9y^2 - 2x - 8 = 0$

51. $x^2 + 9y^2 - 4x - 36y - 41 = 0$

52. $16x^2 + y^2 + 160x - 10y + 361 = 0$

53. $4x^2 + 5y^2 + 32x - 20y + 64 = 0$

54. $2x^2 + 3y^2 - 8x - 30y + 65 = 0$

55. $8x^2 + 5y^2 - 16x + 10y - 27 = 0$

56. $7x^2 + 2y^2 + 28x - 16y + 46 = 0$

57. $36x^2 + 16y^2 - 36x - 32y - 119 = 0$

58. $16x^2 + 100y^2 + 64x - 300y - 111 = 0$

59. $x^2 + 4y^2 - 20y + 21 = 0$

60. $9x^2 + y^2 + 12x - 2y - 4 = 0$

Given general form determine the intercepts.

61. $5x^2 + 4y^2 - 20x + 24y + 36 = 0$

62. $4x^2 + 3y^2 - 8x + 6y - 5 = 0$

63. $6x^2 + y^2 - 12x + 4y + 4 = 0$

64. $8x^2 + y^2 - 6y - 7 = 0$

65. $5x^2 + 2y^2 - 20x - 8y + 18 = 0$

66. $2x^2 + 3y^2 - 4x - 5y + 1 = 0$

Determine the area of the ellipse. (The area of an ellipse is given by the formula $A = \pi ab$, where *a* and *b* are the lengths of the major radius and the minor radius.)

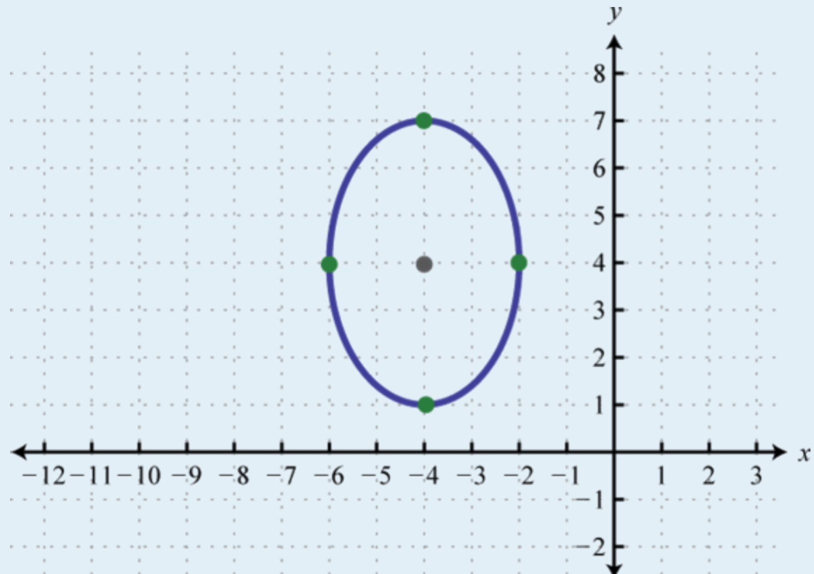
67. $\frac{(x-10)^2}{25} + \frac{(y+3)^2}{5} = 1$

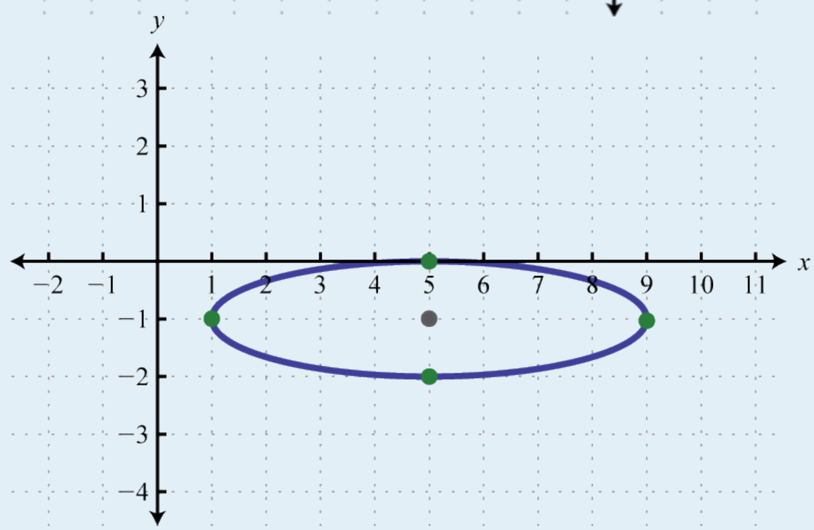
68. $\frac{(x+1)^2}{18} + \frac{y^2}{36} = 1$

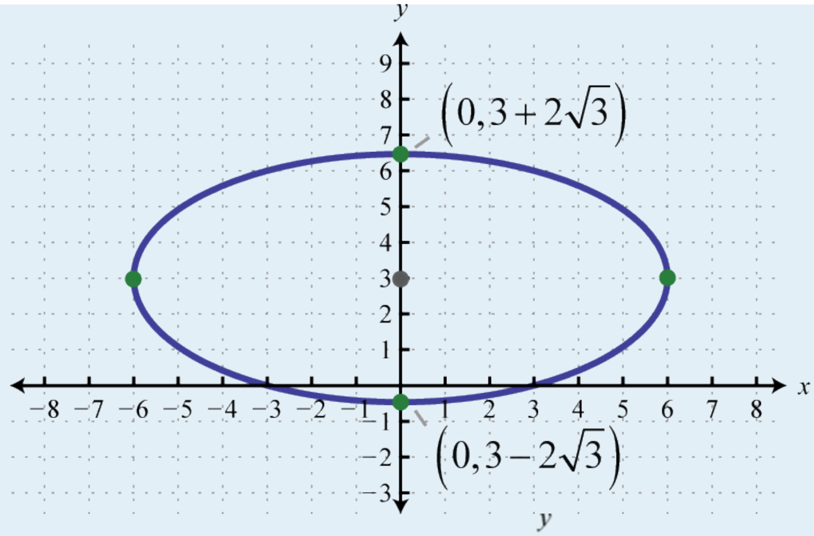
69. $7x^2 + 3y^2 - 14x + 36y + 94 = 0$

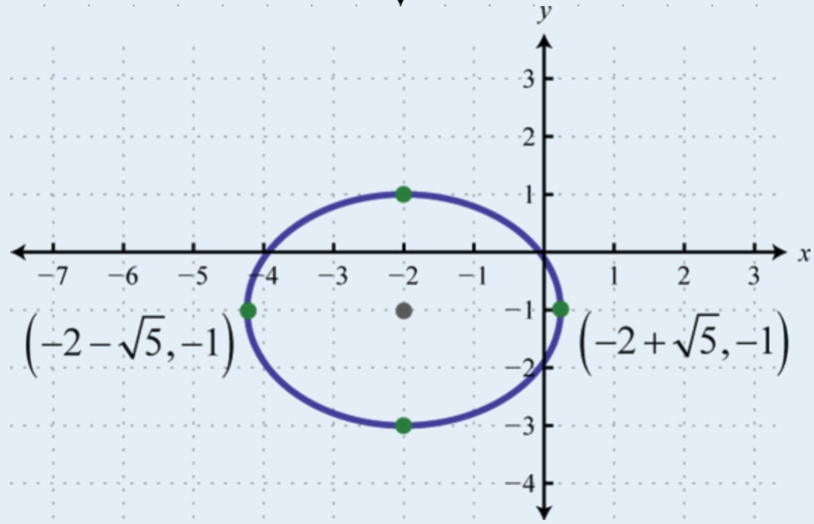
70. $4x^2 + 8y^2 + 20x - 8y + 11 = 0$

Given the graph of an ellipse, determine its equation in general form.

71. 

72. 

73. 

74. 

PART C: DISCUSSION BOARD

75. Explain why a circle can be thought of as a very special ellipse.

76. Make up your own equation of an ellipse, write it in general form and graph it.

77. Do all ellipses have intercepts? What are the possible numbers of intercepts for an ellipse? Explain.

78. Research and discuss real-world examples of ellipses.

ANSWERS

1. Center: $(1, -2)$; orientation: vertical; major radius: 7 units; minor radius: 2 units; $a = 2; b = 7$

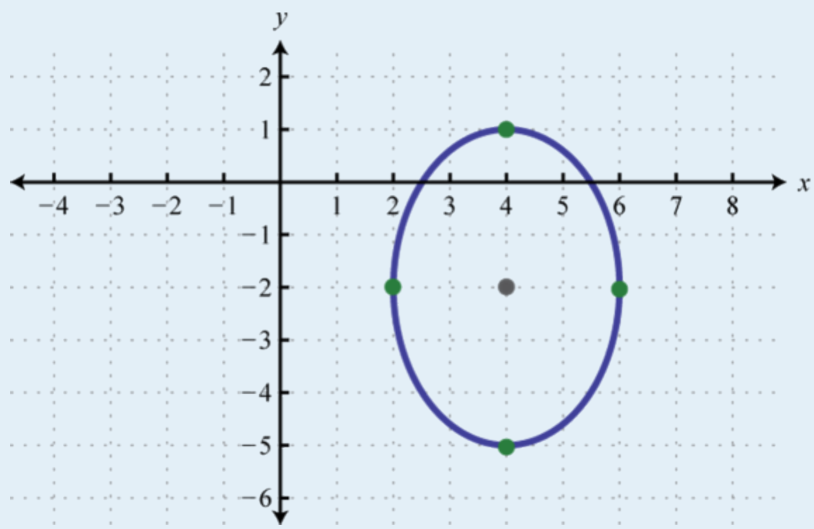
3. Center: $(0, -9)$; orientation: horizontal; major radius: $\sqrt{3}$ units; minor radius: 1 unit; $a = \sqrt{3}; b = 1$

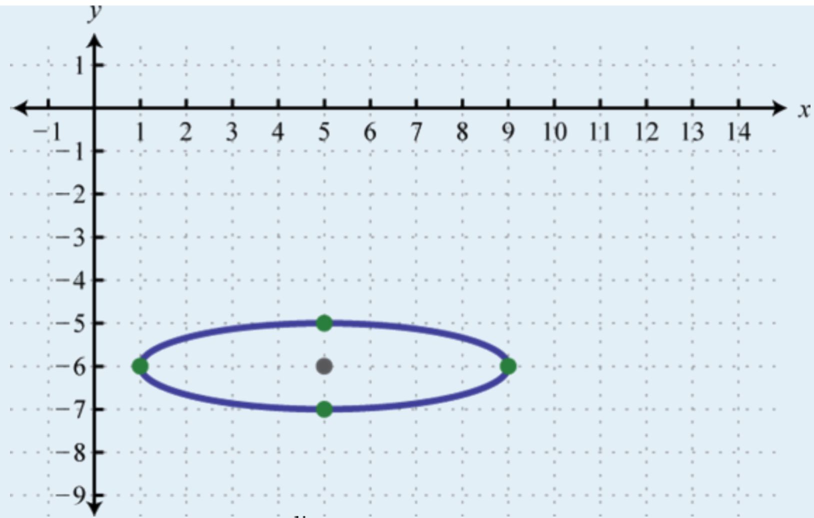
5. Center: $(-5, -5)$; orientation: horizontal; major radius: 3 units; minor radius: 2 units; $a = 3; b = 2$

7. $\frac{(x-3)^2}{25} + \frac{(y-4)^2}{4} = 1$

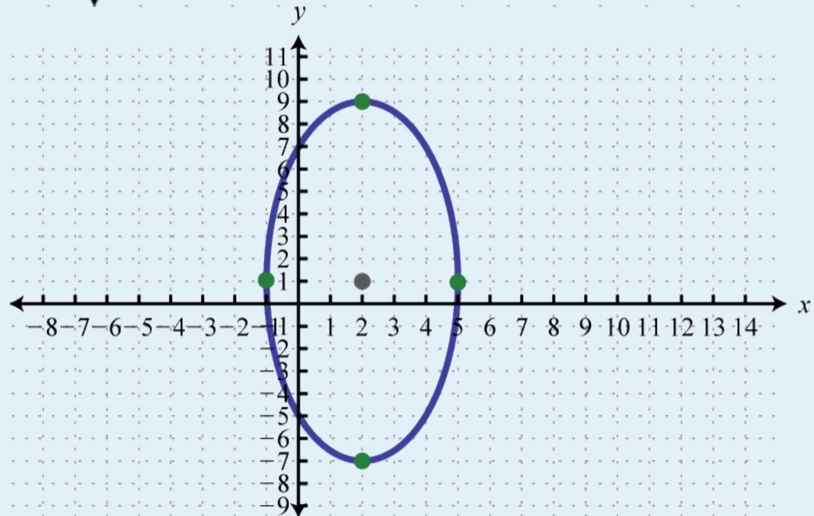
9. $\frac{(x-5)^2}{6} + \frac{(y+1)^2}{12} = 1$

11. $x^2 + \frac{(y+3)^2}{5} = 1$

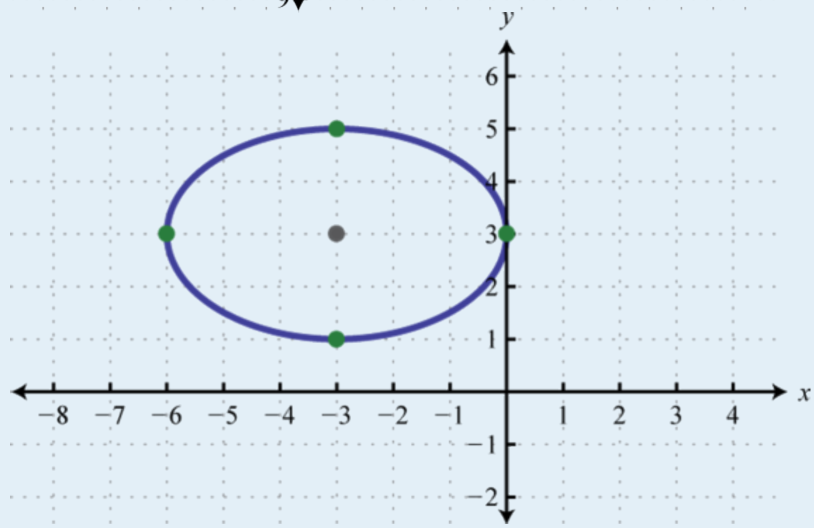
13. 


y
x
1
−1
1 2 3 4 5 6 7 8 9 10 11 12 13 14
−1 −2 −3 −4 −5 −6 −7 −8 −9

15.

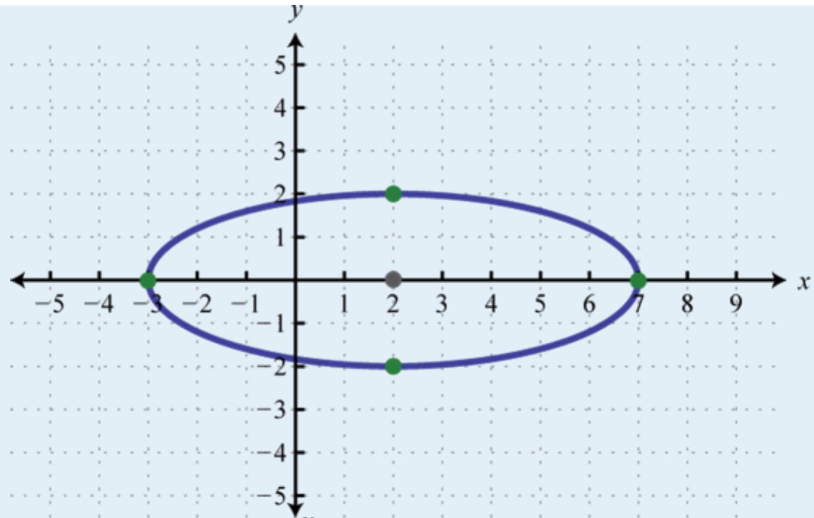

y
x
11 10 9 8 7 6 5 4 3 2 1
−8 −7 −6 −5 −4 −3 −2 −1
1 2 3 4 5 6 7 8 9 10 11 12 13 14
−1 −2 −3 −4 −5 −6 −7 −8 −9

17.


y
x
6 5 4 3 2 1
−8 −7 −6 −5 −4 −3 −2 −1
1 2 3 4
−1 −2

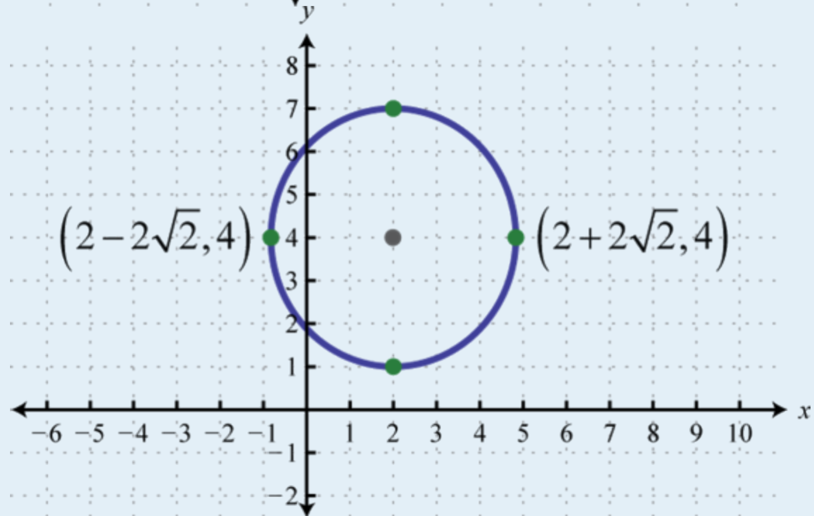
19.

21.



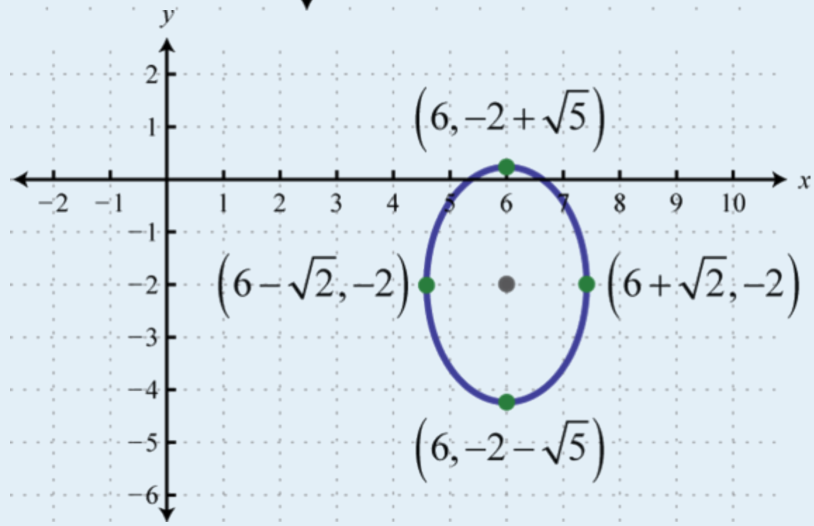
x
y

23.

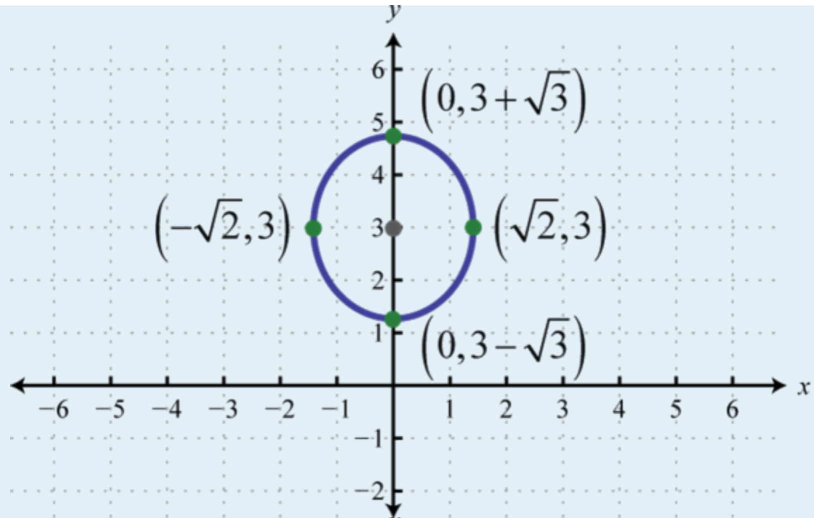


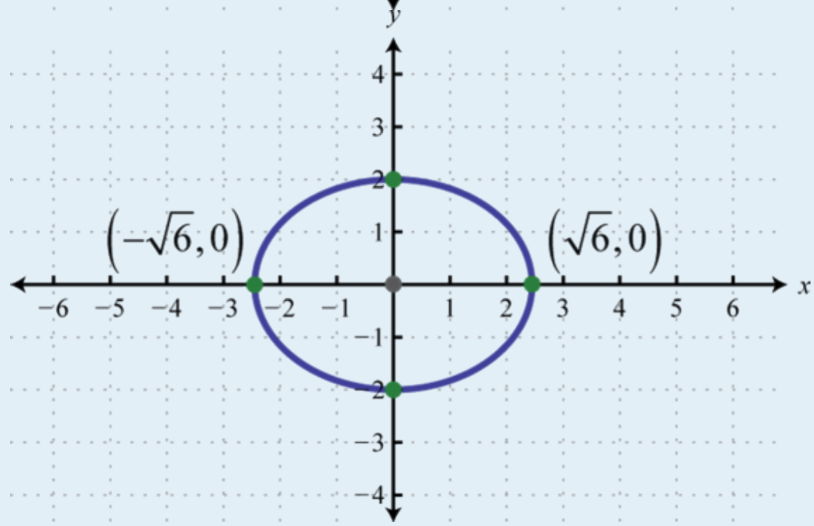
x
y
$\left(2-2\sqrt{2},4\right)$
$\left(2+2\sqrt{2},4\right)$

25.



x
y
$\left(6,-2+\sqrt{5}\right)$
$\left(6-\sqrt{2},-2\right)$
$\left(6+\sqrt{2},-2\right)$
$\left(6,-2-\sqrt{5}\right)$

27. 

29. 

31. x-intercepts: $\left(\frac{9\pm2\sqrt{5}}{3}, 0\right)$; y-intercepts: none

33. x-intercepts: $(2, 0)$; y-intercepts: $(0, -6)$

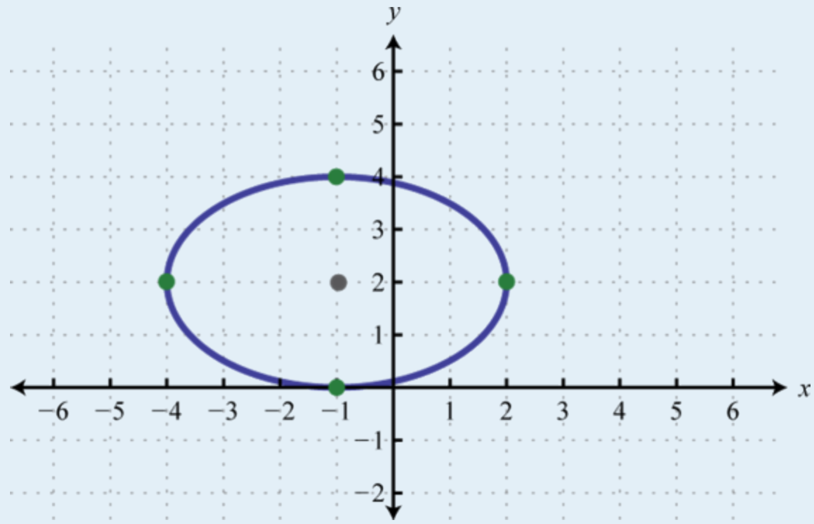
35. x-intercepts: none; y-intercepts: $\left(0, 4 \pm \sqrt{10}\right)$

37. x-intercepts: $\left(\pm\sqrt{2}, 0\right)$; y-intercepts: $\left(0, \pm\sqrt{5}\right)$

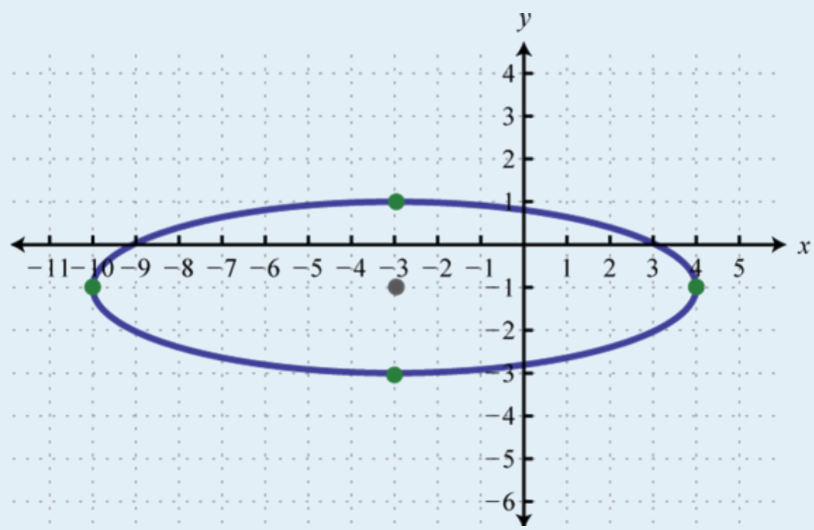
39. $\frac{x^2}{25} + \frac{y^2}{36} = 1$

41. $\frac{(x+4)^2}{16} + \frac{(y+2)^2}{4} = 1$

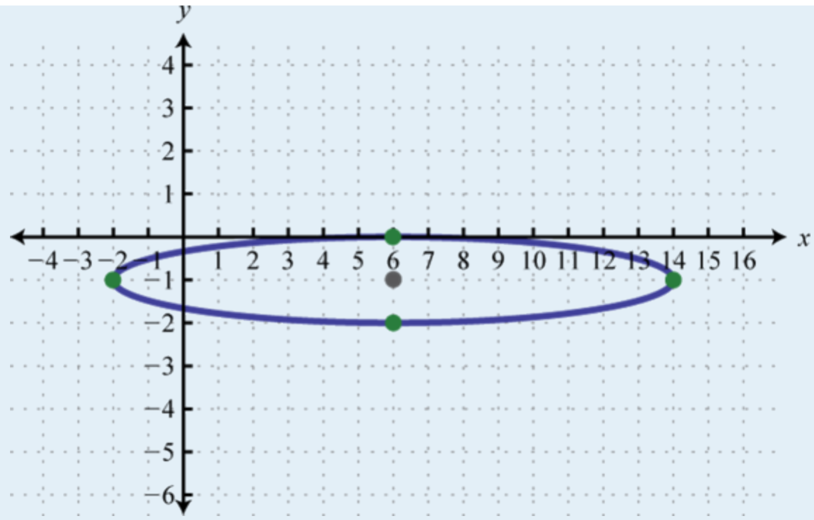
43. $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1;$



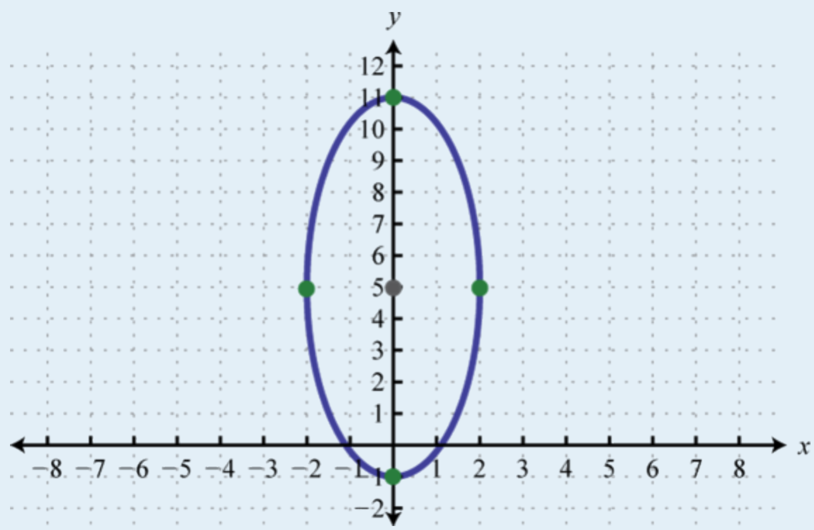
45. $\frac{(x+3)^2}{49} + \frac{(y+1)^2}{4} = 1;$



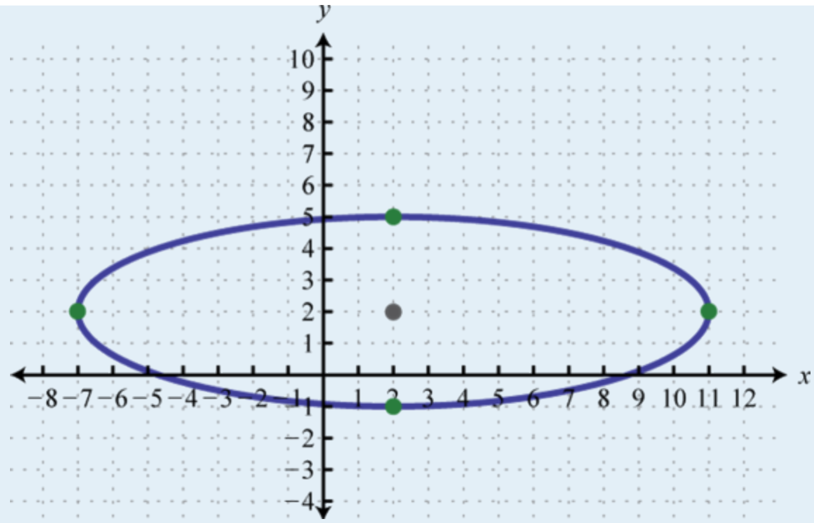
47. $\frac{(x-6)^2}{64} + (y+1)^2 = 1;$



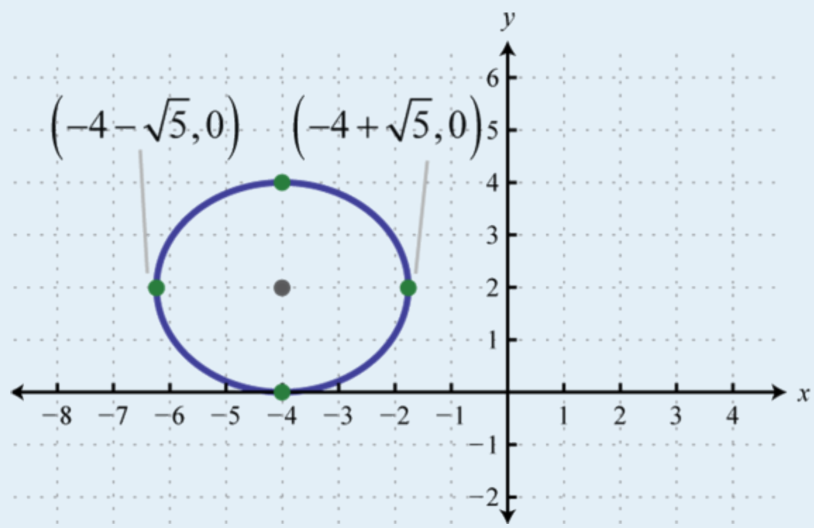
49. $\frac{x^2}{4} + \frac{(y-5)^2}{36} = 1;$



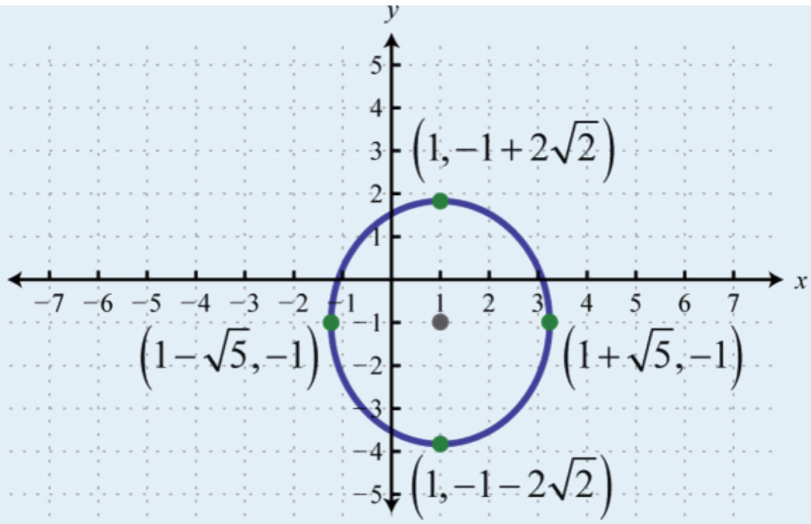
51. $\frac{(x-2)^2}{81} + \frac{(y-2)^2}{9} = 1;$



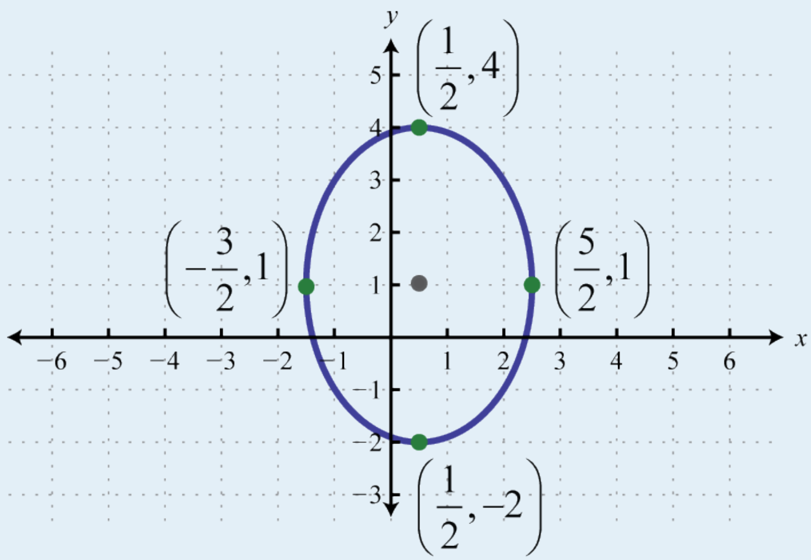
53. $\frac{(x+4)^2}{5} + \frac{(y-2)^2}{4} = 1;$



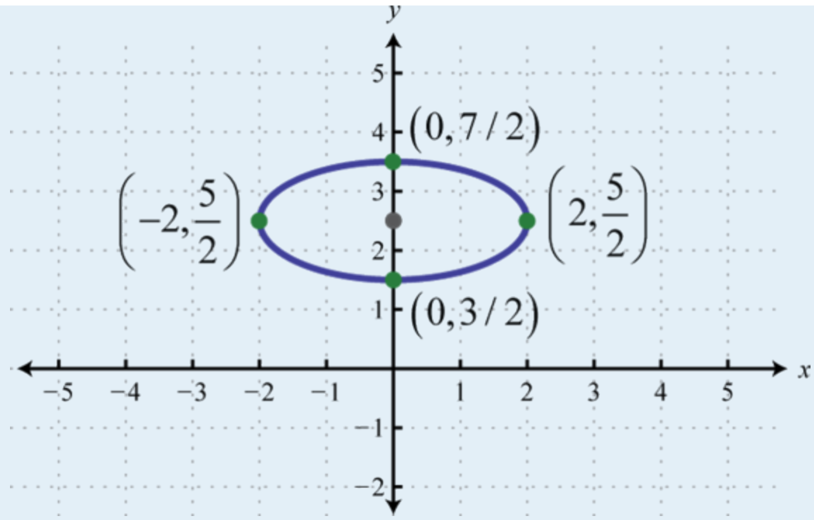
55. $\frac{(x-1)^2}{5} + \frac{(y+1)^2}{8} = 1;$



57. $\frac{\left(x-\frac{1}{2}\right)^2}{4}+\frac{(y-1)^2}{9}=1;$



59. $\frac{x^2}{4}+\left(y-\frac{5}{2}\right)^2=1;$



61. x-intercepts: none; y-intercepts: $(0, -3)$

63. x-intercepts: $\left(\frac{3\pm\sqrt{3}}{3}, 0\right)$; y-intercepts: $(0, -2)$

65. x-intercepts: $\left(\frac{10\pm\sqrt{10}}{5}, 0\right)$; y-intercepts: none

67. $5\pi\sqrt{5}$ square units

69. $\pi\sqrt{21}$ square units

71. $9x^2 + 4y^2 + 72x - 32y + 172 = 0$

73. $x^2 + 3y^2 - 18y - 9 = 0$

75. Answer may vary

77. Answer may vary

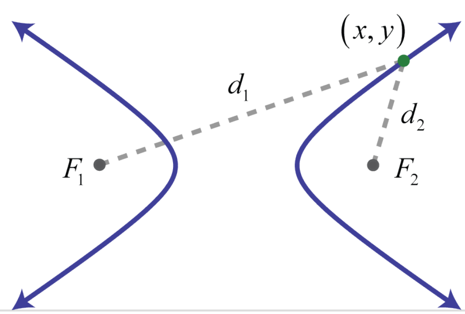
8.4 Hyperbolas

LEARNING OBJECTIVES

1. Graph a hyperbola in standard form.
2. Determine the equation of a hyperbola given its graph.
3. Rewrite the equation of a hyperbola in standard form.
4. Identify a conic section given its equation.

The Hyperbola in Standard Form

A **hyperbola**[23] is the set of points in a plane whose distances from two fixed points, called foci, has an absolute difference that is equal to a positive constant. In other words, if points F_1 and F_2 are the foci and d is some given positive constant then (x, y) is a point on the hyperbola if $d = \ |d_1 - d_2|$as pictured below:

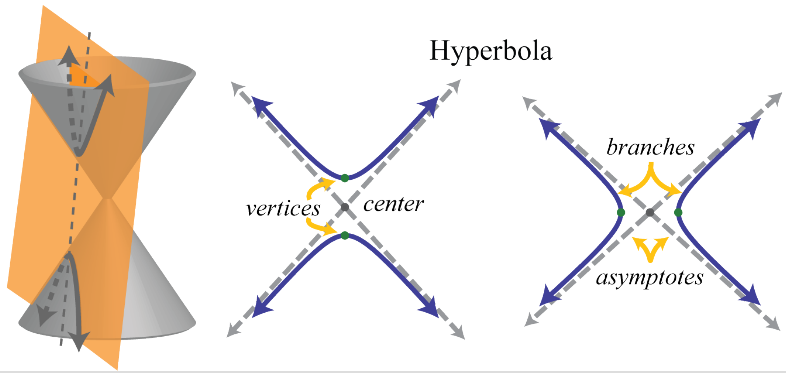


In addition, a hyperbola is formed by the intersection of a cone with an oblique plane that intersects the base. It consists of two separate curves, called **branches**[24]. Points on the separate branches of the graph where the distance is at a minimum are called **vertices.**[25] The midpoint between a hyperbola's vertices is its center. Unlike a parabola, a hyperbola is asymptotic to certain lines drawn through the center. In this section, we will focus on graphing hyperbolas that open left and right or upward and downward.

23. The set of points in a plane whose distances from two fixed points, called foci, has an absolute difference that is equal to a positive constant.

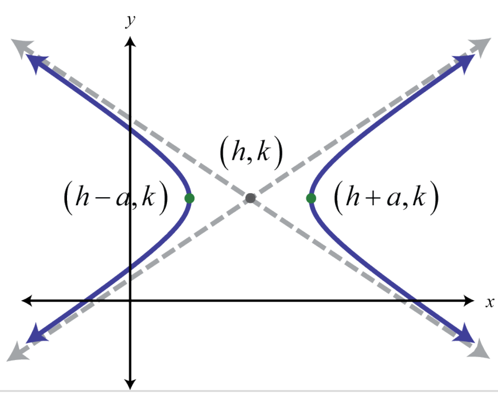
24. The two separate curves of a hyperbola.

25. Points on the separate branches of a hyperbola where the distance is a minimum.



The asymptotes are drawn dashed as they are not part of the graph; they simply indicate the end behavior of the graph. The equation of a **hyperbola opening left and right in standard form**[26] follows:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

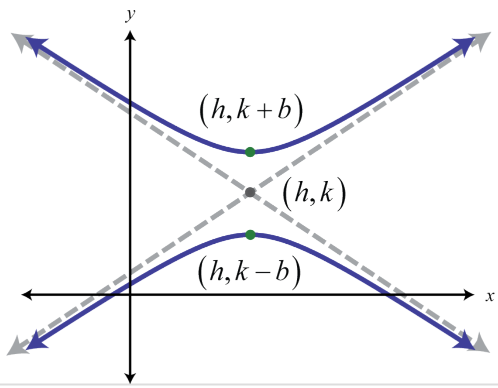


Here the center is (h, k) and the vertices are $(h \pm a, k)$ .The equation of a **hyperbola opening upward and downward in standard form**[27] follows:

$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

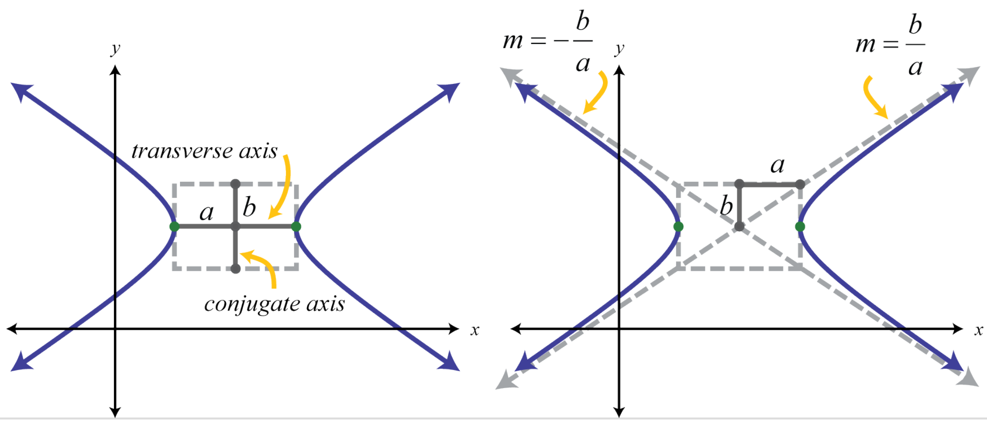
26. The equation of a hyperbola written in the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.The center is (h, k), a defines the transverse axis, and b defines the conjugate axis.

27. The equation of a hyperbola written in the form $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$.The center is (h, k), b defines the transverse axis, and a defines the conjugate axis.



Here the center is (h, k) and the vertices are $(h, k \pm b)$.

The asymptotes are essential for determining the shape of any hyperbola. Given standard form, the asymptotes are lines passing through the center (h, k) with slope $m = \pm \frac{b}{a}$ .To easily sketch the asymptotes we make use of two special line segments through the center using *a* and *b*. Given any hyperbola, the **transverse axis**[28] is the line segment formed by its vertices. The **conjugate axis**[29] is the line segment through the center perpendicular to the transverse axis as pictured below:



The rectangle defined by the transverse and conjugate axes is called the **fundamental rectangle**[30]. The lines through the corners of this rectangle have slopes $m = \pm \frac{b}{a}$ .These lines are the asymptotes that define the shape of the hyperbola. Therefore, given standard form, many of the properties of a hyperbola are apparent.

28. The line segment formed by the vertices of a hyperbola.

29. A line segment through the center of a hyperbola that is perpendicular to the transverse axis.

30. The rectangle formed using the endpoints of a hyperbolas, transverse and conjugate axes.

Equation	Center	a	b	Opens
$\frac{(x-3)^2}{25} - \frac{(y-5)^2}{16} = 1$	$(3, 5)$	$a = 5$	$b = 4$	Left and right
$\frac{(y-2)^2}{36} - \frac{(x+1)^2}{9} = 1$	$(-1, 2)$	$a = 3$	$b = 6$	Upward and downward
$\frac{(y+2)^2}{3} - (x-5)^2 = 1$	$(5, -2)$	$a = 1$	$b = \sqrt{3}$	Upward and downward
$\frac{x^2}{49} - \frac{(y+4)^2}{8} = 1$	$(0, -4)$	$a = 7$	$b = 2\sqrt{2}$	Left and right

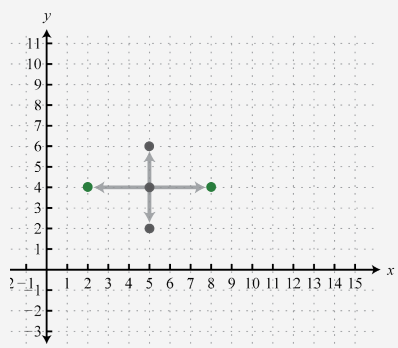
The graph of a hyperbola is completely determined by its center, vertices, and asymptotes.

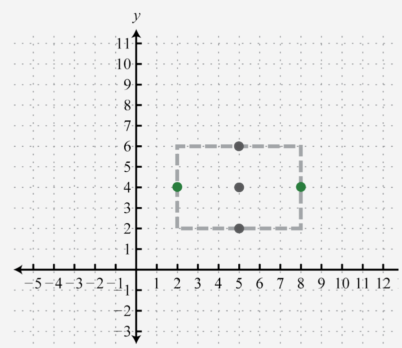
Example 1

Graph: $\frac{(x-5)^2}{9} - \frac{(y-4)^2}{4} = 1.$

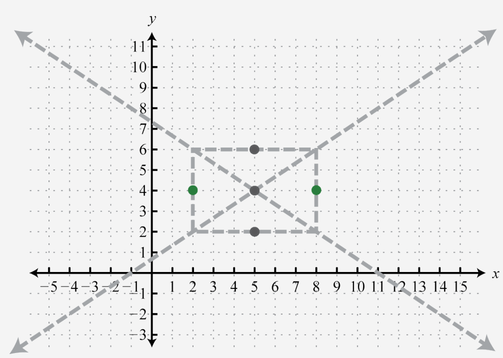
Solution:

In this case, the expression involving x has a positive leading coefficient; therefore, the hyperbola opens left and right. Here $a = \sqrt{9} = 3$ and $b = \sqrt{4} = 2.$ From the center $(5, 4)$, mark points 3 units left and right as well as 2 units up and down. Connect these points with a rectangle as follows:



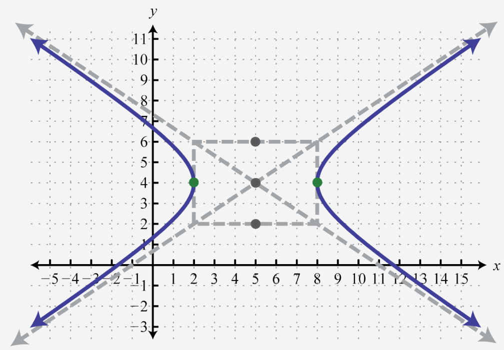


The lines through the corners of this rectangle define the asymptotes.



Use these dashed lines as a guide to graph the hyperbola opening left and right passing through the vertices.

Answer:

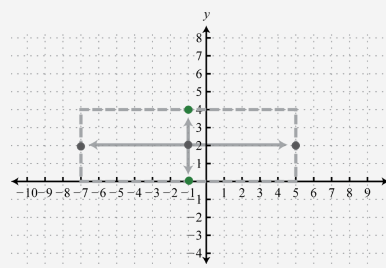


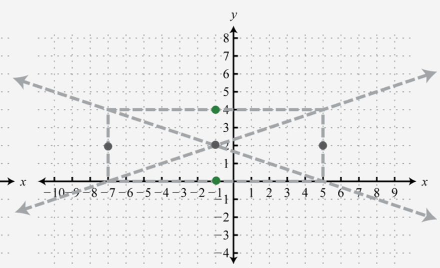
Example 2

Graph: $\frac{(y-2)^2}{4} - \frac{(x+1)^2}{36} = 1.$

Solution:

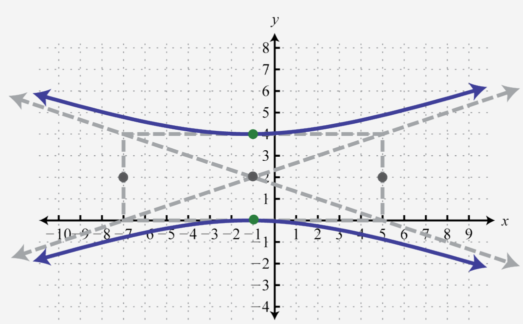
In this case, the expression involving *y* has a positive leading coefficient; therefore, the hyperbola opens upward and downward. Here $a = \sqrt{36} = 6$ and $b = \sqrt{4} = 2$. From the center $(-1, 2)$ mark points 6 units left and right as well as 2 units up and down. Connect these points with a rectangle. The lines through the corners of this rectangle define the asymptotes.





Use these dashed lines as a guide to graph the hyperbola opening upward and downward passing through the vertices.

Answer:



Note: When given a hyperbola opening upward and downward, as in the previous example, it is a common error to interchange the values for the center, *h* and *k*. This is the case because the quantity involving the variable y usually appears first in standard form. Take care to ensure that the y-value of the center comes from the quantity involving the variable y and that the x-value of the center is obtained from the quantity involving the variable x.

As with any graph, we are interested in finding the x- and y-intercepts.

Example 3

Find the intercepts: $\frac{(y-2)^2}{4} - \frac{(x+1)^2}{36} = 1$.

Solution:

To find the *x*-intercepts set $y = 0$ and solve for *x*.

$$\begin{aligned} \frac{(0-2)^2}{4} - \frac{(x+1)^2}{36} &= 1 \\ 1 - \frac{(x+1)^2}{36} &= 1 \\ -\frac{(x+1)^2}{36} &= 0 \\ (x+1)^2 &= 0 \\ x+1 &= 0 \\ x &= -1 \end{aligned}$$

Therefore there is only one *x*-intercept, $(-1, 0)$. To find the *y*-intercept set $x = 0$ and solve for *y*.

$$\frac{(y-2)^2}{4} - \frac{(0+1)^2}{36} = 1$$
$$\frac{(y-2)^2}{4} - \frac{1}{36} = 1$$
$$\frac{(y-2)^2}{4} = \frac{37}{36}$$
$$\frac{(y-2)}{2} = \pm\frac{\sqrt{37}}{6}$$
$$y-2 = \pm\frac{\sqrt{37}}{3}$$
$$y = 2 \pm \frac{\sqrt{37}}{3} = \frac{6 \pm \sqrt{37}}{3}$$

Therefore there are two y-intercepts, $\left(0, \frac{6-\sqrt{37}}{3}\right) \approx (0, -0.03)$ and $\left(0, \frac{6+\sqrt{37}}{3}\right) \approx (0, 4.03)$. Take a moment to compare these to the sketch of the graph in the previous example.

Answer: x-intercept: $(-1, 0)$; y-intercepts: $\left(0, \frac{6-\sqrt{37}}{3}\right)$ and $\left(0, \frac{6+\sqrt{37}}{3}\right)$.

Consider the hyperbola centered at the origin,

$$9x^2 - 5y^2 = 45$$

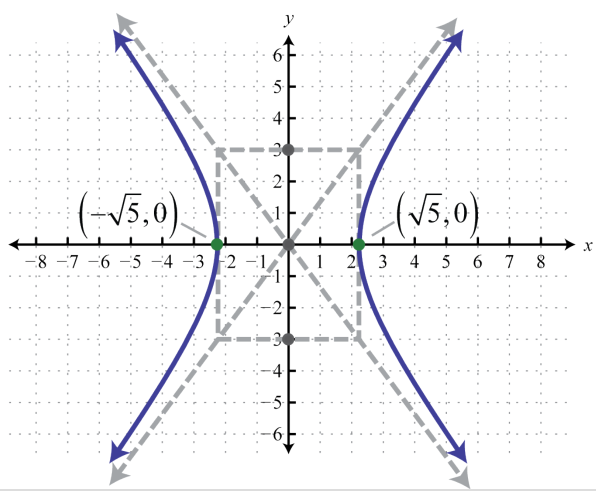
Standard form requires one side to be equal to 1. In this case, we can obtain standard form by dividing both sides by 45.

$$\frac{9x^2 - 5y^2}{45} = \frac{45}{45}$$
$$\frac{9x^2}{45} - \frac{5y^2}{45} = \frac{45}{45}$$
$$\frac{x^2}{5} - \frac{y^2}{9} = 1$$

This can be written as follows:

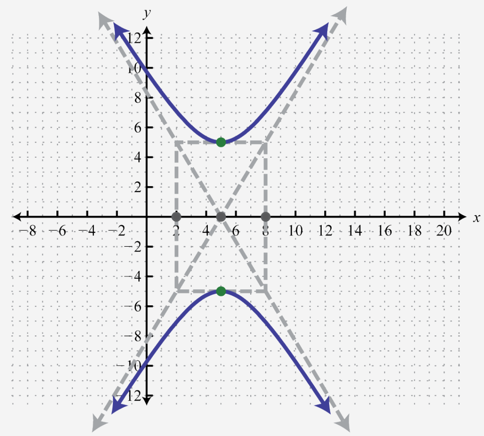
$$\frac{(x-0)^2}{5} - \frac{(y-0)^2}{9} = 1$$

In this form, it is clear that the center is $(0, 0)$, $a = \sqrt{5}$, and $b = 3$. The graph follows.



Try this! Graph: $\frac{y^2}{25} - \frac{(x-5)^2}{9} = 1$.

Answer:



(click to see video)

The Hyperbola in General Form

We have seen that the graph of a hyperbola is completely determined by its center, vertices, and asymptotes; which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of a **hyperbola in general form**[31] follows:

$$px^2 - qy^2 + cx + dy + e = 0 \quad \textit{Hyperbola opens left and right.}$$
$$qy^2 - px^2 + cx + dy + e = 0 \quad \textit{Hyperbola opens upward and downward.}$$

where $p, q > 0$. The steps for graphing a hyperbola given its equation in general form are outlined in the following example.

31. The equation of a hyperbola written in the form $px^2 - qy^2 + cx + dy + e = 0$ or $qy^2 - px^2 - cx + dy + e = 0$ where $p, q > 0$.

Example 4

Graph: $4x^2 - 9y^2 + 32x - 54y - 53 = 0$.

Solution:

Begin by rewriting the equation in standard form.

- **Step 1:** Group the terms with the same variables and move the constant to the right side. Factor so that the leading coefficient of each grouping is 1.

$$\begin{aligned} 4x^2 - 9y^2 + 32x - 54y - 53 &= 0 \\ \left(4x^2 + 32x + \_\_\_\right) + \left(-9y^2 - 54y + \_\_\_\right) &= 53 \\ 4\left(x^2 + 8x + \_\_\_\right) - 9\left(y^2 + 6y + \_\_\_\right) &= 53 \end{aligned}$$

- **Step 2:** Complete the square for each grouping. In this case, for the terms involving x use $\left(\frac{8}{2}\right)^2 = 4^2 = 16$ and for the terms involving y use $\left(\frac{6}{2}\right)^2 = (3)^2 = 9$. The factor in front of each grouping affects the value used to balance the equation on the right,

$$4\left(x^2 + 8x {\color{blue}+ 16}\right) - 9\left(y^2 + 6y {\color{green}+ 9}\right) = 53 {\color{blue}+ 64} {\color{green}- 81}$$

Because of the distributive property, adding 16 inside of the first grouping is equivalent to adding $4 \cdot 16 = 64$. Similarly, adding 9 inside of the second grouping is equivalent to adding $-9 \cdot 9 = -81$. Now factor and then divide to obtain 1 on the right side.

$$4(x+4)^2 - 9(y+3)^2 = 36$$

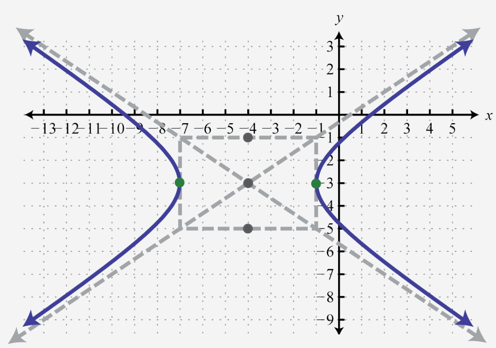
$$\frac{4(x+4)^2 - 9(y+3)^2}{36} = \frac{36}{36}$$

$$\frac{4(x+4)^2}{36} - \frac{9(y+3)^2}{36} = \frac{36}{36}$$

$$\frac{(x+4)^2}{9} - \frac{(y+3)^2}{4} = 1$$

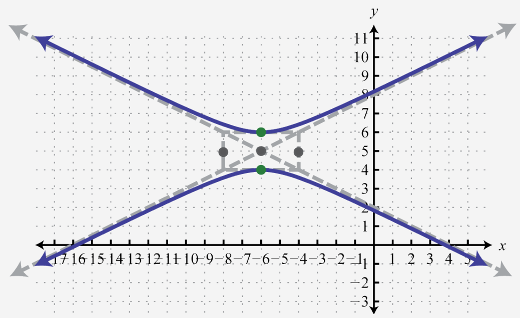
- **Step 3:** Determine the center, a, and b, and then use this information to sketch the graph. In this case, the center is $(-4, -3)$, $a = \sqrt{9} = 3$, and $b = \sqrt{4} = 2$. Because the leading coefficient of the expression involving x is positive and the coefficient of the expression involving y is negative, we graph a hyperbola opening left and right.

Answer:



Try this! Graph: $4y^2 - x^2 - 40y - 12x + 60 = 0$.

Answer:



(click to see video)

Identifying the Conic Sections

In this section, the challenge is to identify a conic section given its equation in general form. To distinguish between the conic sections, use the exponents and coefficients. If the equation is quadratic in only one variable and linear in the other, then its graph will be a parabola.

Parabola: $a > 0$	
$y = a(x-h)^2 + k$ $y = ax^2 + bx + c$	$x = a(y-k)^2 + h$ $x = ay^2 + by + c$

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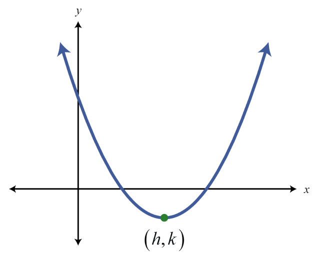
<td>
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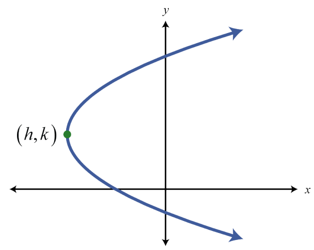
<td>
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<td>Parabola: $a < 0$</td>
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<tr>
<td>$$y=a(x-h)^2+k$$ $$y=ax^2+bx+c$$</td>
<td>$$x=a(y-k)^2+h$$ $$x=ay^2+by+c$$</td>
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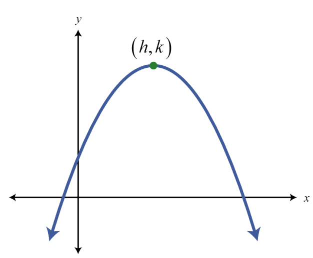
<td>
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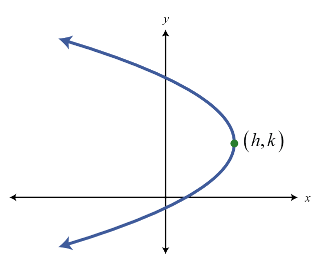
<td>
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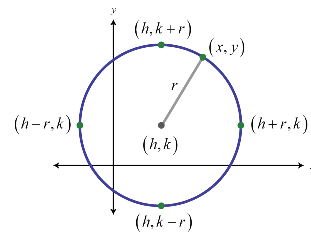
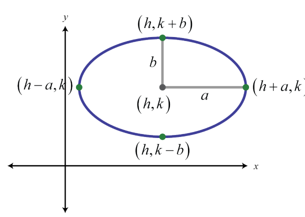
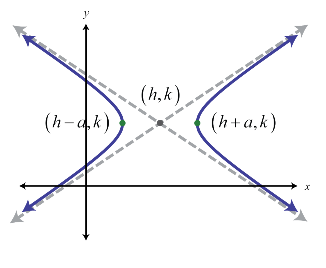
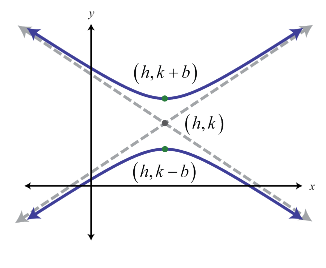
If the equation is quadratic in both variables, where the coefficients of the squared terms are the same, then its graph will be a circle.

Circle:	
$(x-h)^2+(y-k)^2=r^2$ $x^2+y^2+cx+dy+e=0$	

If the equation is quadratic in both variables where the coefficients of the squared terms are different but have the same sign, then its graph will be an ellipse.

Ellipse: $a,b>0$ and $p,q>0$	
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $px^2+qy^2+cx+dy+e=0$	

If the equation is quadratic in both variables where the coefficients of the squared terms have different signs, then its graph will be a hyperbola.

Hyperbola: $a, b > 0$ and $p, q > 0$	
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $px^2 - qy^2 + cx + dy + e = 0$	$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ $qy^2 - px^2 + cx + dy + e = 0$
	

Example 5

Identify the graph of each equation as a parabola, circle, ellipse, or hyperbola.

a. $4x^2 + 4y^2 - 1 = 0$
b. $3x^2 - 2y^2 - 12 = 0$
c. $x - y^2 - 6y + 11 = 0$

Solution:

a. The equation is quadratic in both x and y where the leading coefficients for both variables is the same, 4.

$$\begin{aligned} 4x^2 + 4y^2 - 1 &= 0 \\ 4x^2 + 4y^2 &= 1 \\ x^2 + y^2 &= \frac{1}{4} \end{aligned}$$

This is an equation of a circle centered at the origin with radius 1/2.

b. The equation is quadratic in both x and y where the leading coefficients for both variables have different signs.

$$\begin{aligned} 3x^2 - 2y^2 - 12 &= 0 \\ \frac{3x^2 - 2y^2}{12} &= \frac{12}{12} \\ \frac{x^2}{4} - \frac{y^2}{6} &= 1 \end{aligned}$$

This is an equation of a hyperbola opening left and right centered at the origin.

c. The equation is quadratic in y only.

$$\begin{aligned} x - y^2 + 6y - 11 &= 0 \\ x &= y^2 - 6y + \quad\quad + 11 \\ x &= (y^2 - 6y + 9) + 11 - 9 \\ x &= (y-3)^2 + 2 \end{aligned}$$

This is an equation of a parabola opening right with vertex $(2, 3)$.

Answer:

a. Circle
b. Hyperbola
c. Parabola

KEY TAKEAWAYS

- The graph of a hyperbola is completely determined by its center, vertices, and asymptotes.
- The center, vertices, and asymptotes are apparent if the equation of a hyperbola is given in standard form: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1.$
- To graph a hyperbola, mark points *a* units left and right from the center and points *b* units up and down from the center. Use these points to draw the fundamental rectangle; the lines through the corners of this rectangle are the asymptotes. If the coefficient of x^2 is positive, draw the branches of the hyperbola opening left and right through the points determined by *a*. If the coefficient of y^2 is positive, draw the branches of the hyperbola opening up and down through the points determined by *b*.
- The orientation of the transverse axis depends the coefficient of x^2 and y^2.
- If the equation of a hyperbola is given in general form $px^2 - qy^2 + cx + dy + e = 0$ or $qy^2 - px^2 + cx + dy + e = 0$ where $p, q > 0$, group the terms with the same variables, and complete the square for both groupings to obtain standard form.
- We recognize the equation of a hyperbola if it is quadratic in both *x* and *y* where the coefficients of the squared terms are opposite in sign.

TOPIC EXERCISES

PART A: THE HYPERBOLA IN STANDARD FORM

Given the equation of a hyperbola in standard form, determine its center, which way the graph opens, and the vertices.

1. $\frac{(x-6)^2}{16} - \frac{(y+4)^2}{9} = 1$
2. $\frac{(y-3)^2}{25} - \frac{(x+1)^2}{64} = 1$
3. $\frac{(y+9)^2}{5} - x^2 = 1$
4. $\frac{(x-5)^2}{12} - y^2 = 1$
5. $4(y+10)^2 - 25(x+1)^2 = 100$
6. $9(x-1)^2 - 5(y+10)^2 = 45$

Determine the standard form for the equation of a hyperbola given the following information.

7. Center $(2, 7)$, $a = 6$, $b = 3$, opens left and right.
8. Center $(-9, 1)$, $a = 7$, $b = 2$, opens up and down.
9. Center $(10, -3)$, $a = \sqrt{7}$, $b = 5\sqrt{2}$, opens up and down.
10. Center $(-7, -2)$, $a = 3\sqrt{3}$, $b = \sqrt{5}$, opens left and right.
11. Center $(0, -8)$, $a = \sqrt{2}$, $b = 1$,opens up and down.
12. Center $(0, 0)$, $a = 2\sqrt{6}$, $b = 4$, opens left and right.

Graph.

13. $\frac{(x-3)^2}{9} - \frac{(y+1)^2}{16} = 1$

14. $\frac{(x+3)^2}{4} - \frac{(y-1)^2}{25} = 1$

15. $\frac{(x-2)^2}{16} - \frac{(y+3)^2}{1} = 1$

16. $\frac{(y+2)^2}{9} - \frac{(x+2)^2}{36} = 1$

17. $\frac{(y-1)^2}{4} - \frac{(x-2)^2}{16} = 1$

18. $(y+2)^2 - \frac{(x+3)^2}{9} = 1$

19. $4(x+3)^2 - 9(y-3)^2 = 36$

20. $16x^2 - 4(y-1)^2 = 64$

21. $4(y-1)^2 - 25x^2 = 100$

22. $9y^2 - 16x^2 = 144$

23. $\frac{(x-2)^2}{12} - \frac{(y-4)^2}{9} = 1$

24. $\frac{(x+2)^2}{4} - \frac{(y-1)^2}{8} = 1$

25. $\frac{(y+1)^2}{5} - \frac{(x-3)^2}{2} = 1$

26. $\frac{(y-4)^2}{3} - \frac{(x+6)^2}{18} = 1$

27. $4x^2 - 3(y-3)^2 = 12$

28. $7(x+1)^2 - 2y^2 = 14$

29. $6y^2 - 3x^2 = 18$

30. $10x^2 - 3y^2 = 30$

Find the *x*- and *y*-intercepts.

31. $\frac{(x-1)^2}{9} - \frac{(y-4)^2}{4} = 1$

32. $\frac{(x+4)^2}{16} - \frac{(y-3)^2}{9} = 1$

33. $\frac{(y-1)^2}{4} - \frac{(x+1)^2}{36} = 1$

34. $\frac{(y+2)^2}{4} - \frac{(x-1)^2}{16} = 1$

35. $2x^2 - 3(y-1)^2 = 12$

36. $6(x-5)^2 - 2y^2 = 12$

37. $36x^2 - 2y^2 = 9$

38. $6y^2 - 4x^2 = 2$

39. Find the equation of the hyperbola with vertices $(\pm 2, 3)$ and a conjugate axis that measures 12 units.

40. Find the equation of the hyperbola with vertices $(4, 7)$ and $(4, 3)$ and a conjugate axis that measures 6 units.

PART B: THE HYPERBOLA IN GENERAL FORM

Rewrite in standard form and graph.

41. $4x^2 - 9y^2 + 16x + 54y - 101 = 0$

42. $9x^2 - 25y^2 - 18x - 100y - 316 = 0$

43. $4y^2 - 16x^2 - 64x + 8y - 124 = 0$

44. $9y^2 - 4x^2 - 24x - 72y + 72 = 0$

45. $y^2 - 36x^2 - 72x - 12y - 36 = 0$

46. $9y^2 - x^2 + 8x - 36y + 11 = 0$

47. $36x^2 - 4y^2 + 24y - 180 = 0$

48. $x^2 - 25y^2 - 2x - 24 = 0$

49. $25x^2 - 64y^2 + 200x + 640y - 2{,}800 = 0$

50. $49y^2 - 4x^2 + 40x + 490y + 929 = 0$

51. $3x^2 - 2y^2 + 24x + 8y + 34 = 0$
52. $4x^2 - 8y^2 - 24x + 80y - 196 = 0$
53. $3y^2 - x^2 - 2x - 6y - 16 = 0$
54. $12y^2 - 5x^2 + 40x + 48y - 92 = 0$
55. $4x^2 - 16y^2 + 12x + 16y - 11 = 0$
56. $4x^2 - y^2 - 4x - 2y - 16 = 0$
57. $4y^2 - 36x^2 + 108x - 117 = 0$
58. $4x^2 - 9y^2 + 8x + 6y - 33 = 0$

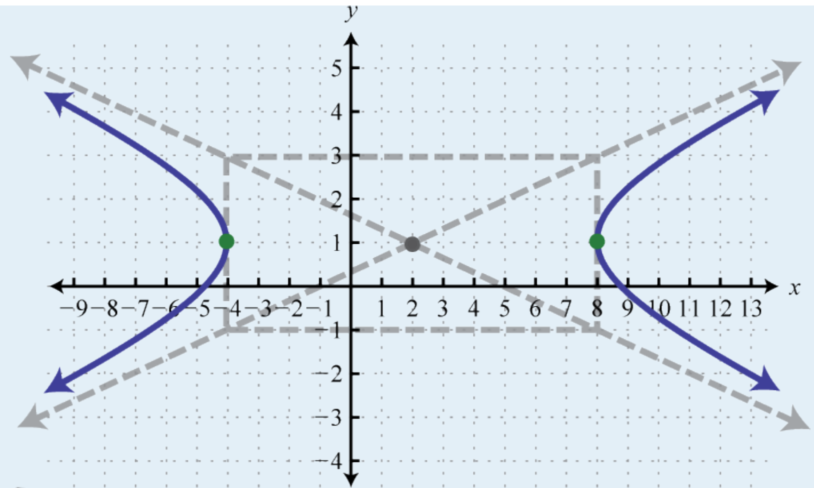
Given the general form, determine the intercepts.

59. $3x^2 - y^2 - 11x - 8y - 4 = 0$
60. $4y^2 - 8x^2 + 2x + 9y - 9 = 0$
61. $x^2 - y^2 + 2x + 2y - 4 = 0$
62. $y^2 - x^2 + 6y - 8x - 16 = 0$
63. $5x^2 - 2y^2 - 4x - 3y = 0$
64. $2x^2 - 3y^2 - 4x - 5y + 1 = 0$

Find the equations of the asymptotes to the given hyperbola.

65. $\frac{(y-5)^2}{9} - \frac{(x+8)^2}{16} = 1.$
66. $\frac{(x+9)^2}{36} - \frac{(y-4)^2}{4} = 1.$
67. $16x^2 - 4y^2 - 24y - 96x + 44 = 0.$
68. $4y^2 - x^2 - 8y - 4x - 4 = 0.$

Given the graph of a hyperbola, determine its equation in general form.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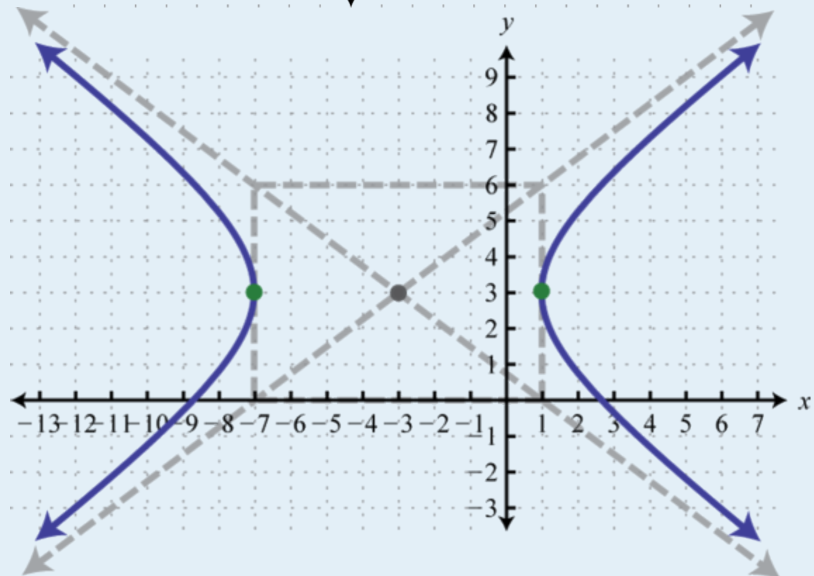
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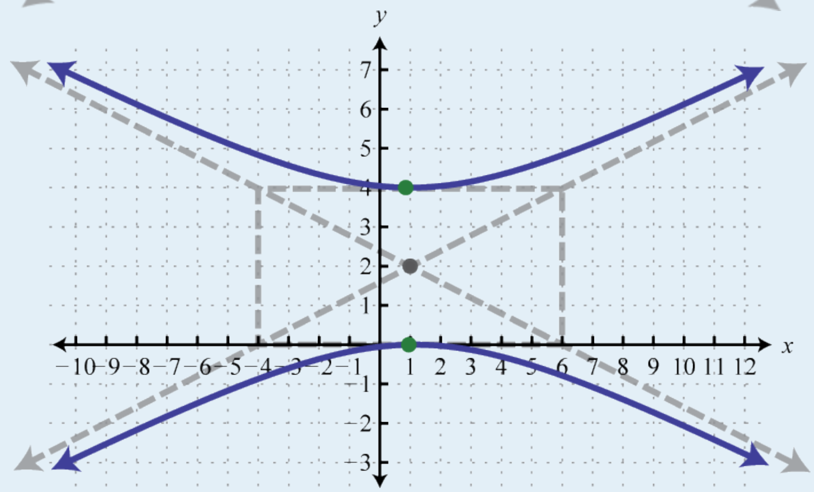
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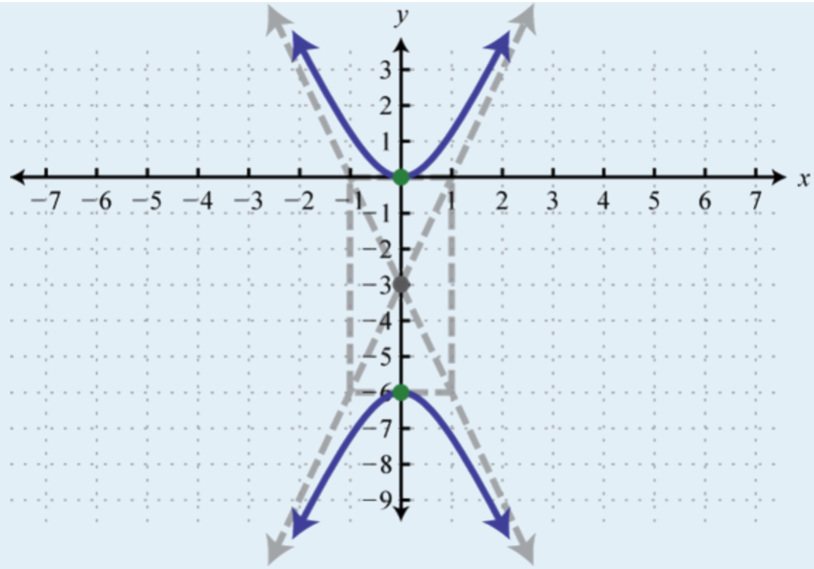
71.

72.



PART C: IDENTIFYING THE CONIC SECTIONS

Identify the following as the equation of a line, parabola, circle, ellipse, or hyperbola.

73. $x^2 + y^2 + 10x - 2y + 23 = 0$

74. $x^2 + y + 2x - 3 = 0$

75. $2x^2 + y^2 - 12x + 14 = 0$

76. $3x - 2y = 24$

77. $x^2 - y^2 + 36 = 0$

78. $4x^2 + 4y^2 - 32 = 0$

79. $x^2 - y^2 - 2x + 2y - 1 = 0$

80. $x - y^2 + 2y + 1 = 0$

81. $3x + 3y + 5 = 0$

82. $8x^2 + 4y^2 - 144x - 12y + 641 = 0$

Identify the conic sections and rewrite in standard form.

83. $x^2 - y - 6x + 11 = 0$

84. $x^2 + y^2 - 12x - 6y + 44 = 0$
85. $x^2 - 2y^2 - 4x - 12y - 18 = 0$
86. $25y^2 - 2x^2 + 36x - 50y - 187 = 0$
87. $7x^2 + 4y^2 - 84x + 16y + 240 = 0$
88. $4x^2 + 4y^2 - 80x + 399 = 0$
89. $4x^2 + 4y^2 + 4x - 32y + 29 = 0$
90. $16x^2 - 4y^2 - 32x + 20y - 25 = 0$
91. $9x - 18y^2 + 12y + 7 = 0$
92. $16x^2 + 12y^2 - 24x - 48y + 9 = 0$

PART D: DISCUSSION BOARD

93. Develop a formula for the equations of the asymptotes of a hyperbola. Share it along with an example on the discussion board.
94. Make up your own equation of a hyperbola, write it in general form, and graph it.
95. Do all hyperbolas have intercepts? What are the possible numbers of intercepts for a hyperbola? Explain.
96. Research and discuss real-world examples of hyperbolas.

ANSWERS

1. Center: $(6, -4)$; $a = 4$; $b = 3$; opens left and right; vertices: $(2, -4)$, $(10, -4)$

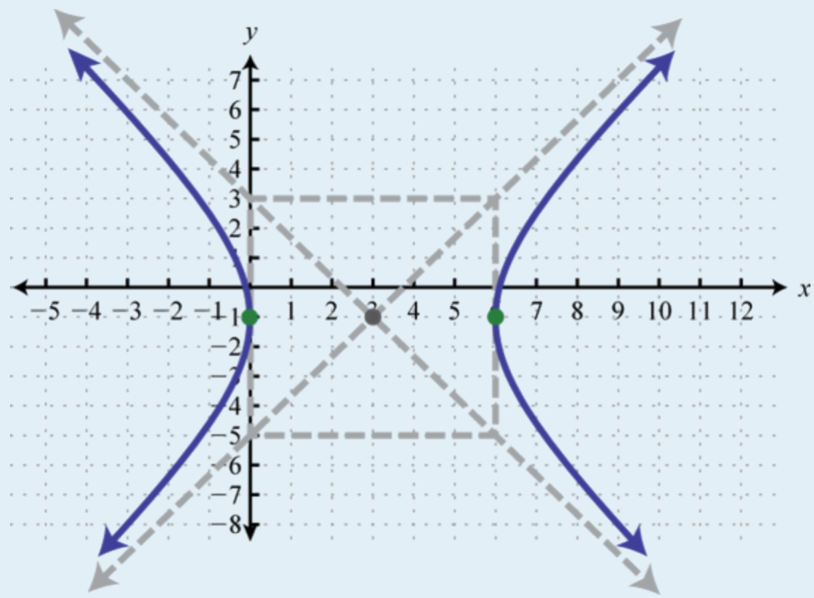
3. Center: $(0, -9)$; $a = 1, b = \sqrt{5}$; opens upward and downward; vertices: $\left(0, -9 - \sqrt{5}\right), \left(0, -9 + \sqrt{5}\right)$

5. Center: $(-1, -10)$; $a = 2, b = 5$; opens upward and downward; vertices: $(-1, -15)$, $(-1, -5)$

7. $\frac{(x-2)^2}{36} - \frac{(y-7)^2}{9} = 1$

9. $\frac{(y+3)^2}{50} - \frac{(x-10)^2}{7} = 1$

11. $\frac{(y+8)^2}{1} - \frac{x^2}{2} = 1$



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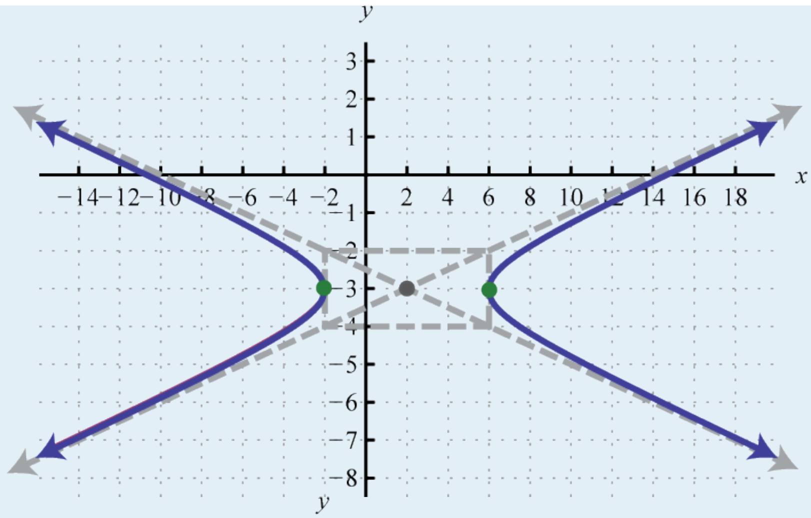
15. 
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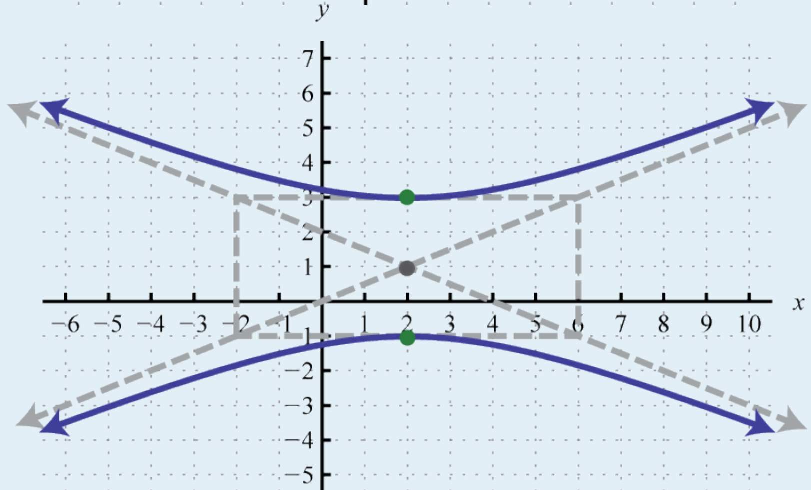
17. 
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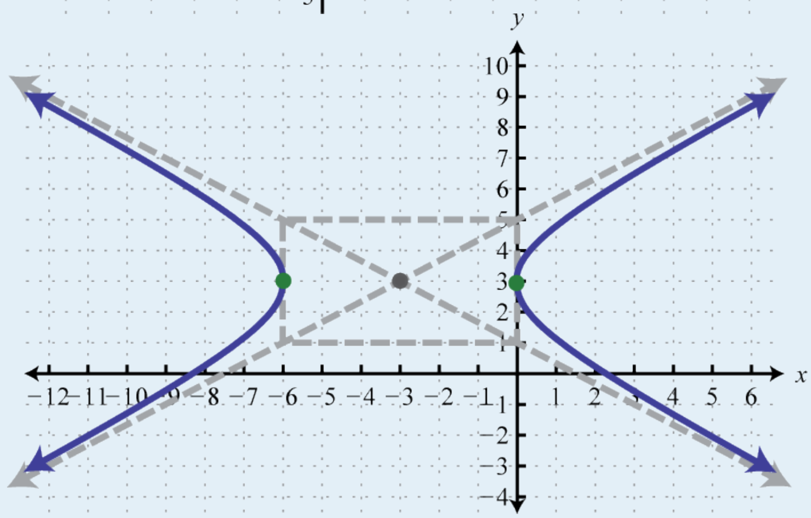
19. 
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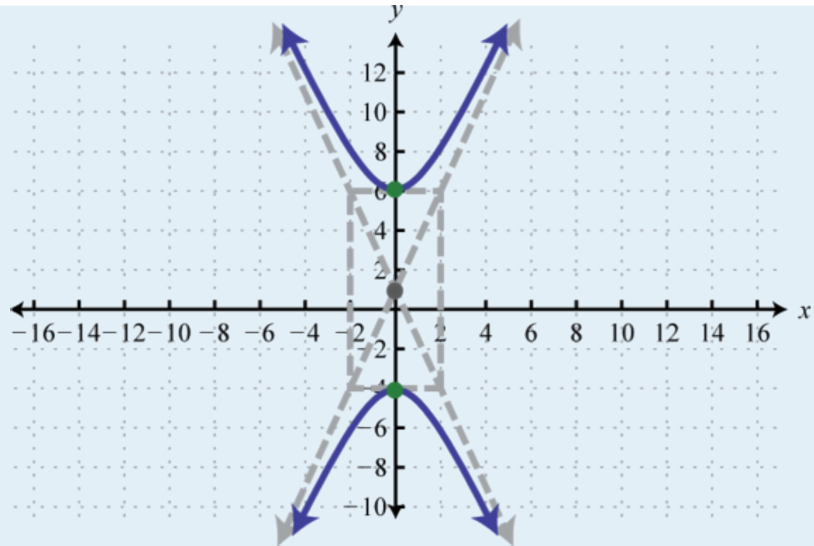

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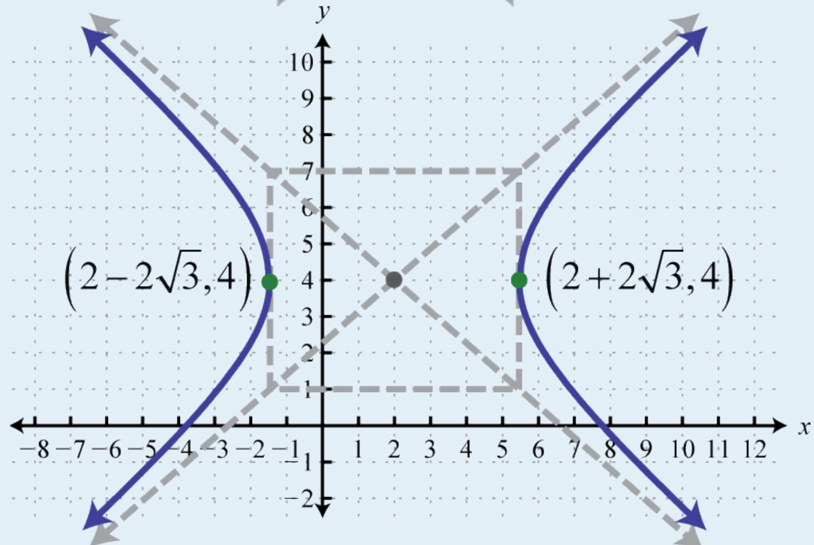
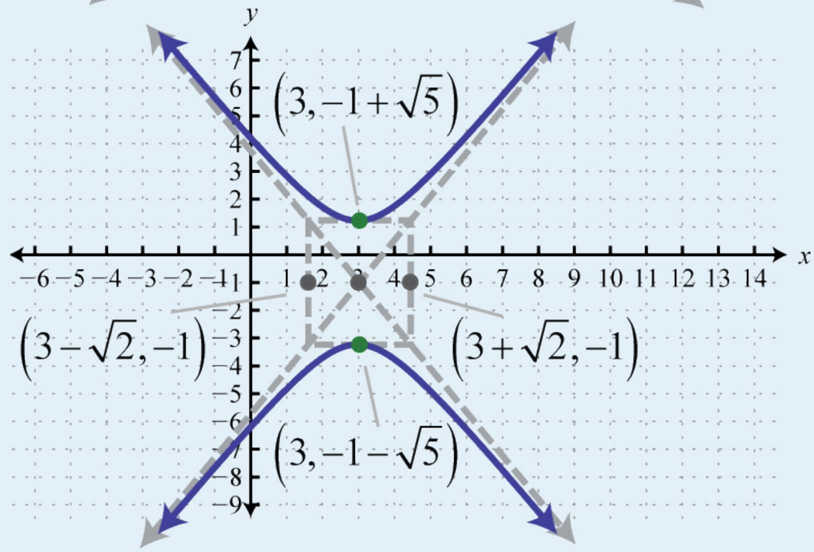
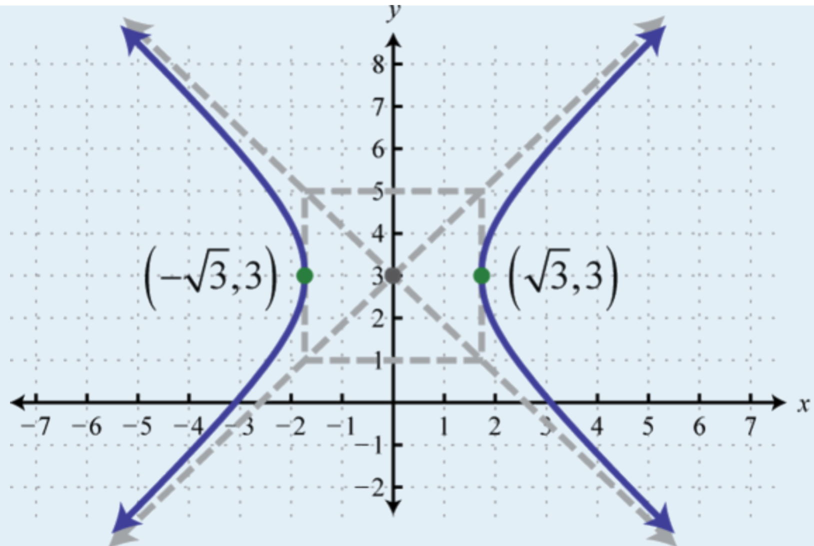
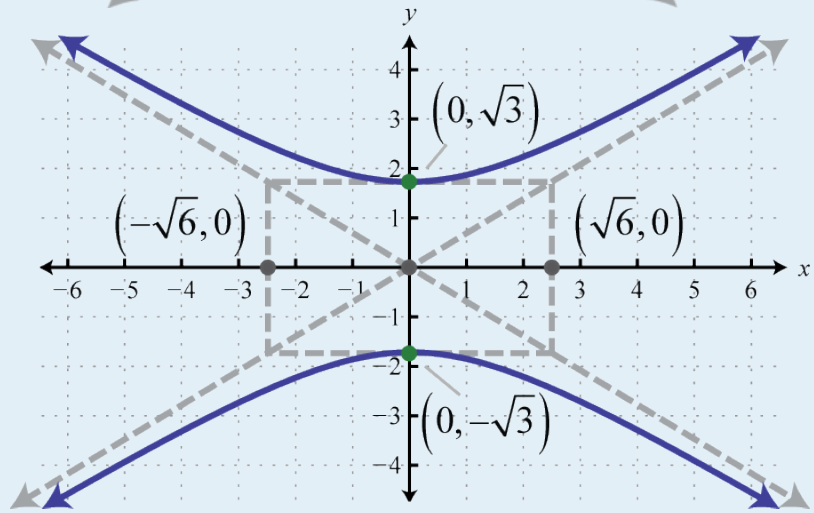

$\left(2-2\sqrt{3},4\right)$
$\left(2+2\sqrt{3},4\right)$

23.


$\left(3,-1+\sqrt{5}\right)$
$\left(3-\sqrt{2},-1\right)$
$\left(3+\sqrt{2},-1\right)$
$\left(3,-1-\sqrt{5}\right)$

25.

27. 

29. 

31. x-intercepts: $\left(1 \pm 3\sqrt{5}, 0\right)$; y-intercepts: none

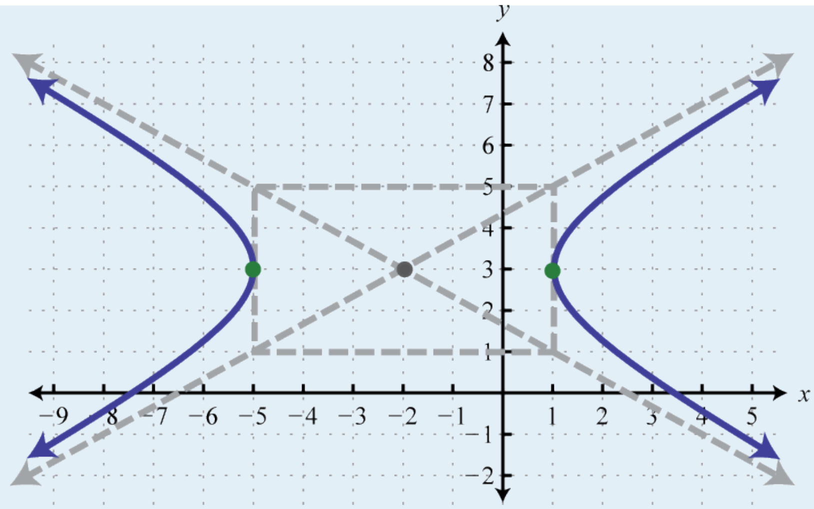
33. x-intercepts: none; y-intercepts: $\left(0, \frac{3\pm\sqrt{37}}{3}\right)$

35. x-intercepts: $\left(\pm \frac{\sqrt{30}}{2}, 0\right)$; y-intercepts: none

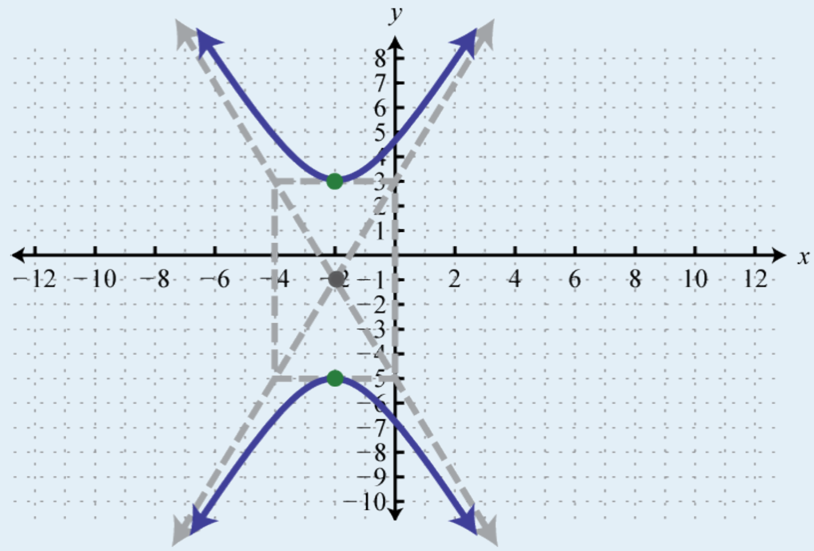
37. x-intercepts: $\left(\pm \frac{1}{2}, 0\right)$; y-intercepts: none

39. $\frac{x^2}{4} - \frac{(y-3)^2}{36} = 1$

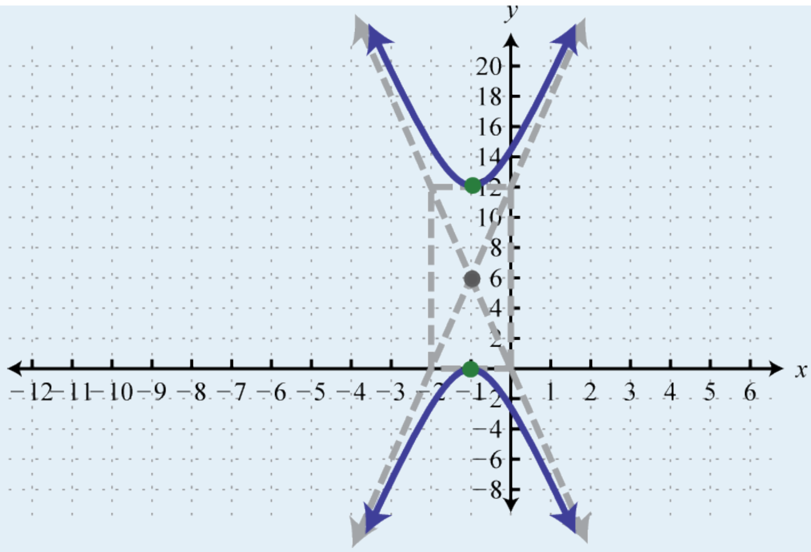
41. $\frac{(x+2)^2}{9} - \frac{(y-3)^2}{4} = 1$;



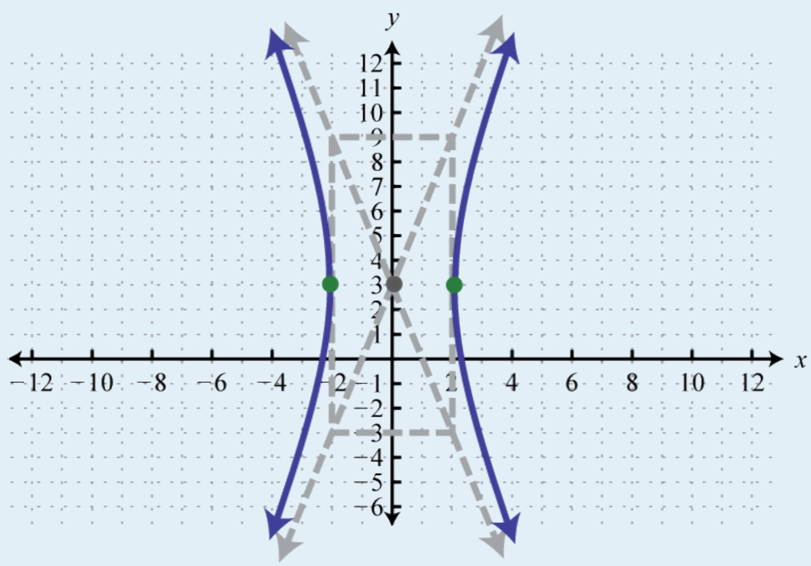
43. $\frac{(y+1)^2}{16} - \frac{(x+2)^2}{4} = 1$;



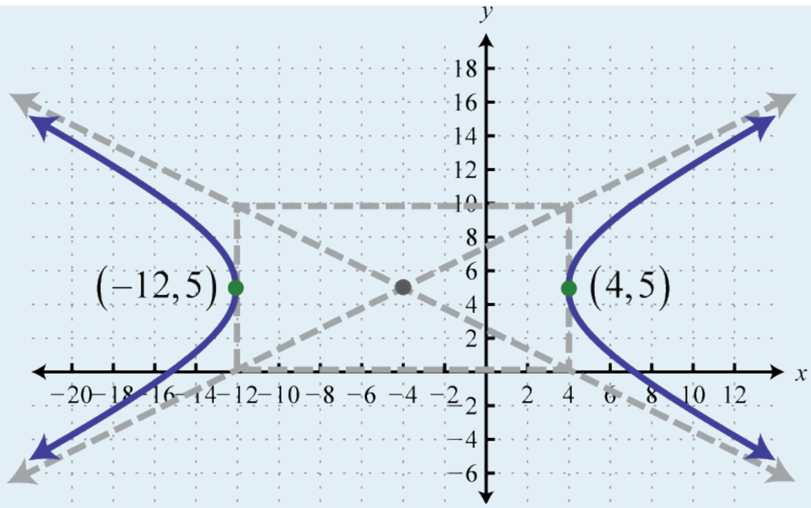
45. $\frac{(y-6)^2}{36} - (x+1)^2 = 1$;



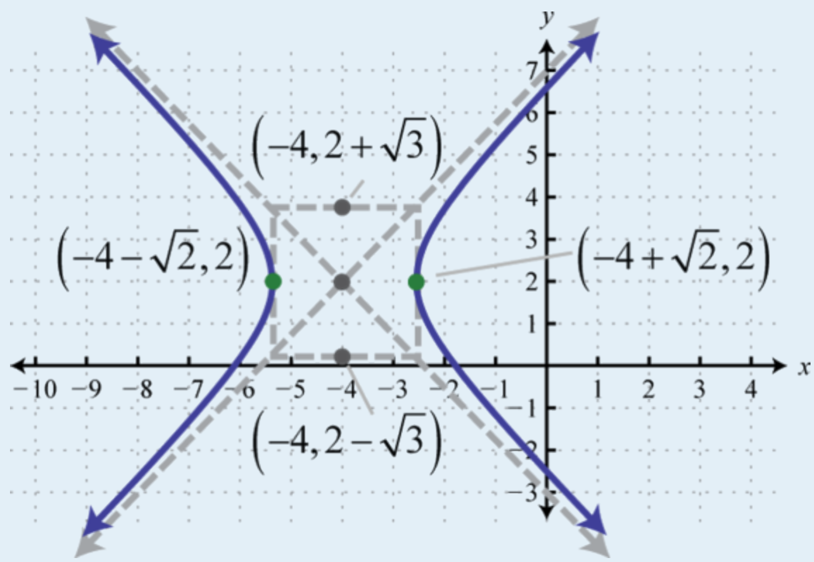
47. $\frac{x^2}{4} - \frac{(y-3)^2}{36} = 1;$



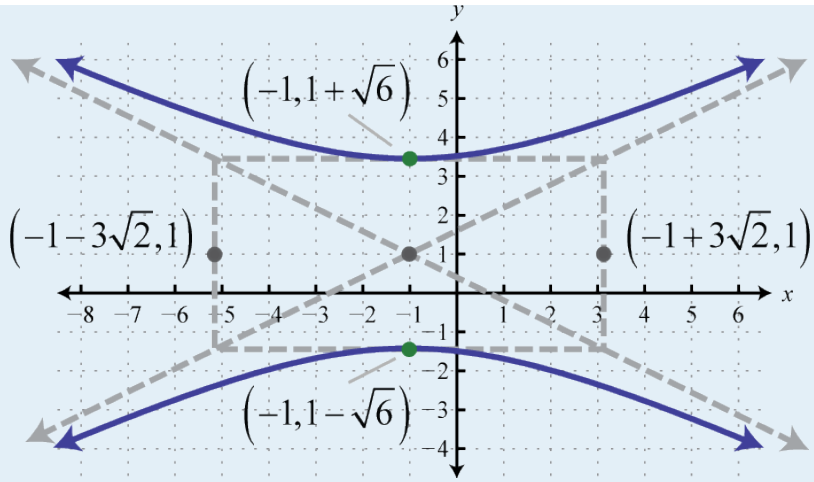
49. $\frac{(x+4)^2}{64} - \frac{(y-5)^2}{25} = 1;$



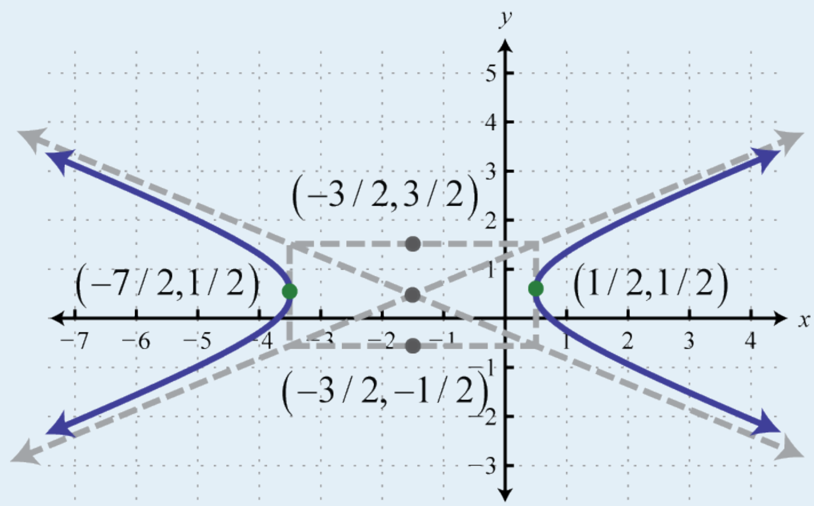
51. $\frac{(x+4)^2}{2} - \frac{(y-2)^2}{3} = 1;$



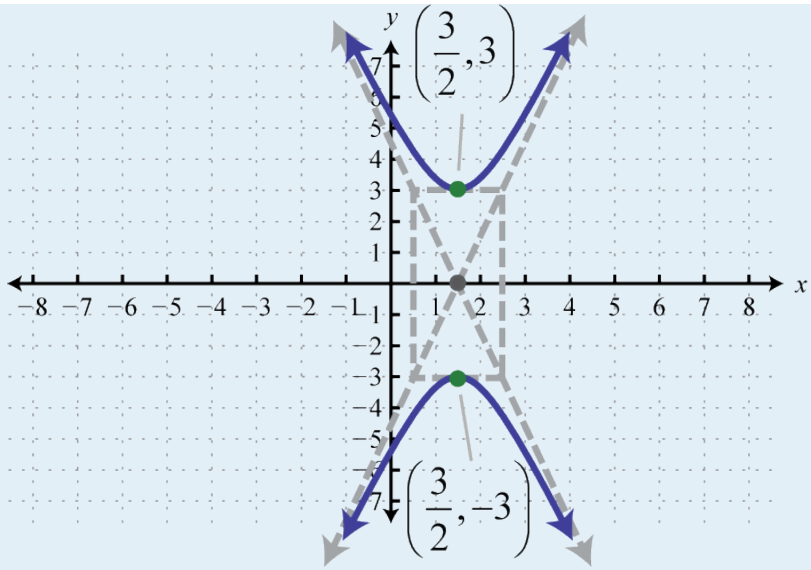
53. $\frac{(y-1)^2}{6} - \frac{(x+1)^2}{18} = 1;$



55. $\frac{\left(x+\frac{3}{2}\right)^2}{4} - \frac{\left(y-\frac{1}{2}\right)^2}{1} = 1;$



57. $\frac{y^2}{9} - \left(x - \frac{3}{2}\right)^2 = 1;$



59. x-intercepts: $\left(-\frac{1}{3}, 0\right), (4, 0)$; y-intercepts: $\left(0, -4 \pm 2\sqrt{3}\right)$

61. x-intercepts: $\left(-1 \pm \sqrt{5}, 0\right)$; y-intercepts: none

63. x-intercepts: $(0, 0), \left(\frac{4}{5}, 0\right)$; y-intercepts: $(0, 0), \left(0, -\frac{3}{2}\right)$

65. $y = -\frac{3}{4}x - 1, y = \frac{3}{4}x + 11$

67. $y = -2x + 3, y = 2x - 9$

69. $x^2 - 9y^2 - 4x + 18y - 41 = 0$

71. $25y^2 - 4x^2 - 100y + 8x - 4 = 0$

73. Circle

75. Ellipse

77. Hyperbola

79. Hyperbola

81. Line

83. Parabola; $y = (x-3)^2 + 2$

85. Hyperbola; $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{2} = 1$

87. Ellipse; $\frac{(x-6)^2}{4}+\frac{(y+2)^2}{7}=1$

89. Circle; $\left(x+\frac{1}{2}\right)^2+(y-4)^2=9$

91. Parabola; $x=2\left(y-\frac{1}{3}\right)^2-1$

93. Answer may vary

95. Answer may vary

8.5 Solving Nonlinear Systems

LEARNING OBJECTIVES

1. Identify nonlinear systems.
2. Solve nonlinear systems using the substitution method.

Nonlinear Systems

A system of equations where at least one equation is not linear is called a **nonlinear system**[32]. In this section we will use the substitution method to solve nonlinear systems. Recall that solutions to a system with two variables are ordered pairs (x, y) that satisfy both equations.

32. A system of equations where at least one equation is not linear.

Example 1

Solve: $\begin{cases} x+2y=0 \\ x^2+y^2=5 \end{cases}$.

Solution:

In this case we begin by solving for x in the first equation.

$$\begin{cases} x+2y=0 \Longrightarrow \quad x=-2y \\ x^2+y^2=5 \end{cases}$$

Substitute $x=-2y$ into the second equation and then solve for y.

$$\begin{aligned} (-2y)^2+y^2 &= 5 \\ 4y^2+y^2 &= 5 \\ 5y^2 &= 5 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

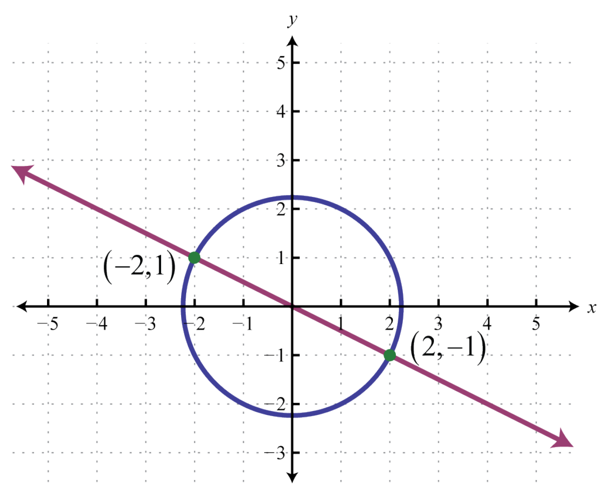
Here there are two answers for y; use $x=-2y$ to find the corresponding x-values.

Using $y = -1$	Using $y = 1$
$x = -2y$ $= -2(-1)$ $= 2$	$x = -2y$ $= -2(1)$ $= -2$

This gives us two ordered pair solutions, $(2, -1)$ and $(-2, 1)$.

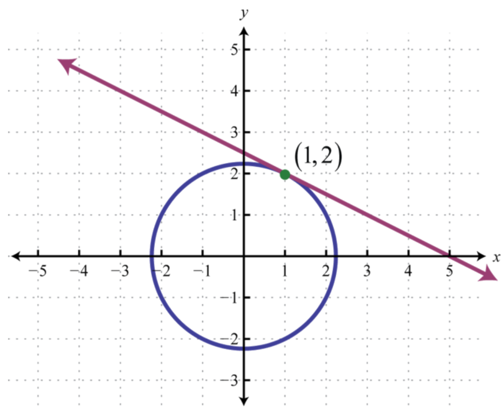
Answer: $(2, -1)$, $(-2, 1)$

In the previous example, the given system consisted of a line and a circle. Graphing these equations on the same set of axes, we can see that the two ordered pair solutions correspond to the two points of intersection.

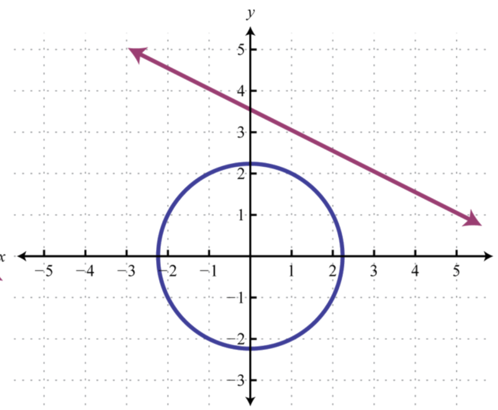


Two solutions.

If we are given a system consisting of a circle and a line, then there are 3 possibilities for real solutions—two solutions as pictured above, one solution, or no solution.



One solution.



No solution.

Example 2

Solve: $\begin{cases} x + y = 3 \\ x^2 + y^2 = 2 \end{cases}$.

Solution:

Solve for y in the first equation.

$$\begin{cases} x + y = 3 \Longrightarrow & y = 3 - x \\ x^2 + y^2 = 2 \end{cases}$$

Next, substitute $y = 3 - x$ into the second equation and then solve for x.

$$\begin{aligned} x^2 + (3 - x)^2 &= 2 \\ x^2 + 9 - 6x + x^2 &= 2 \\ 2x^2 - 6x + 9 &= 2 \\ 2x^2 - 6x + 7 &= 0 \end{aligned}$$

The resulting equation does not factor. Furthermore, using $a = 2$, $b = -6$, and $c = 7$ we can see that the discriminant is negative:

$$\begin{aligned} b^2 - 4ac &= (-6)^2 - 4(2)(7) \\ &= 36 - 56 \\ &= -20 \end{aligned}$$

We conclude that there are no real solutions to this equation and thus no solution to the system.

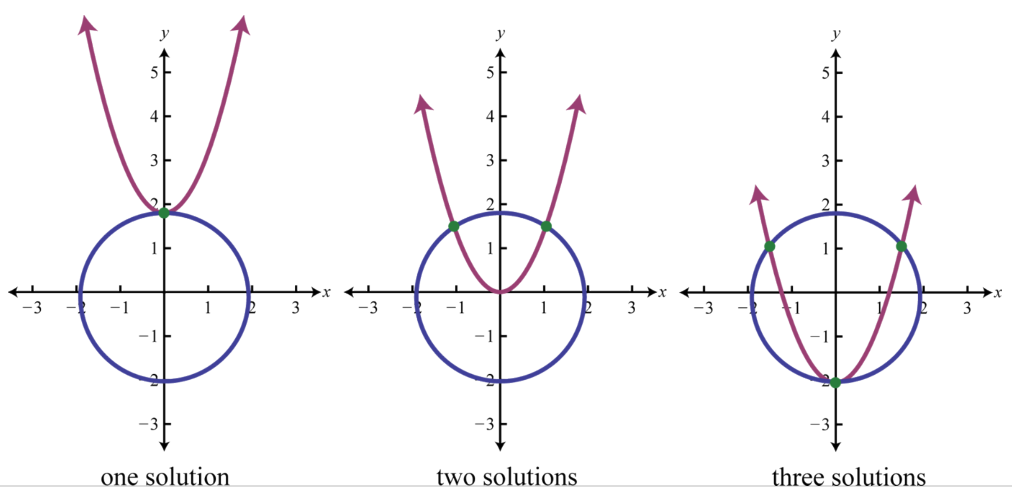
Answer: Ø

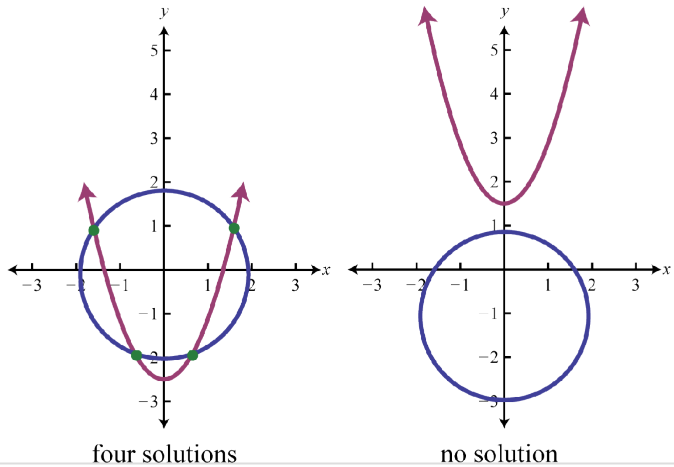
Try this! Solve: $\begin{cases} x - y = 5 \\ x^2 + (y+1)^2 = 8 \end{cases}$.

Answer: $(2, -3)$

(click to see video)

If given a circle and a parabola, then there are 5 possibilities for solutions.





When using the substitution method, we can perform the substitution step using entire algebraic expressions. The goal is to produce a single equation in one variable that can be solved using the techniques learned up to this point in our study of algebra.

Example 3

Solve: $\begin{cases} x^2 + y^2 = 2 \\ y - x^2 = -2 \end{cases}$.

Solution:

We can solve for x^2 in the second equation.

$$\begin{cases} x^2 + y^2 = 2 \\ y - x^2 = -2 \end{cases} \quad \Rightarrow \quad y + 2 = x^2$$

Substitute $x^2 = y + 2$ into the first equation and then solve for y.

$$\begin{aligned} y + 2 + y^2 &= 2 \\ y^2 + y &= 0 \\ y\left(y + 1\right) &= 0 \\ y &= 0 \quad or \quad y = -1 \end{aligned}$$

Back substitute into $x^2 = y + 2$ to find the corresponding x-values.

Using $y = -1$	Using $y = 0$
$x^2 = y + 2$ $x^2 = -1 + 2$ $x^2 = 1$ $x = \pm 1$	$x^2 = y + 2$ $x^2 = 0 + 2$ $x^2 = 2$ $x = \pm\sqrt{2}$

This leads us to four solutions, $(\pm 1, -1)$ and $\left(\pm\sqrt{2}, 0\right)$.

Answer: $(\pm 1, -1)$, $\left(\pm\sqrt{2}, 0\right)$

Example 4

Solve: $\begin{cases} (x-1)^2 - 2y^2 = 4 \\ x^2 + y^2 = 9 \end{cases}.$

Solution:

We can solve for y^2 in the second equation,

$$\begin{cases} (x-1)^2 - 2y^2 = 4 \\ x^2 + y^2 = 9 \Longrightarrow y^2 = 9 - x^2 \end{cases}$$

Substitute $y^2 = 9 - x^2$ into the first equation and then solve for x.

$$\begin{aligned} (x-1)^2 - 2(9 - x^2) &= 4 \\ x^2 - 2x + 1 - 18 + 2x^2 &= 0 \\ 3x^2 - 2x - 21 &= 0 \\ (3x+7)(x-3) &= 0 \\ 3x + 7 &= 0 \quad or \quad x - 3 = 0 \\ x &= -\frac{7}{3} \qquad\quad x = 3 \end{aligned}$$

Back substitute into $y^2 = 9 - x^2$ to find the corresponding y-values.

Using $x = -\frac{7}{3}$	Using $x = 3$
$y^2 = 9 - \left(-\frac{7}{3}\right)^2$ $y^2 = \frac{9}{1} - \frac{49}{9}$ $y^2 = \frac{32}{9}$ $y = \pm\frac{\sqrt{32}}{3} = \pm\frac{4\sqrt{2}}{3}$	$y^2 = 9 - (3)^2$ $y^2 = 0$ $y = 0$

This leads to three solutions, $\left(-\frac{7}{3}, \pm\frac{4\sqrt{2}}{3}\right)$and $(3, 0)$.

Answer: $(3, 0)$, $\left(-\frac{7}{3}, \pm\frac{4\sqrt{2}}{3}\right)$

Example 5

Solve: $\begin{cases} x^2 + y^2 = 2 \\ xy = 1 \end{cases}$.

Solution:

Solve for y in the second equation.

$$\begin{cases} x^2 + y^2 = 2 \\ xy = 1 \Longrightarrow y = \dfrac{1}{x} \end{cases}$$

Substitute $y = \frac{1}{x}$ into the first equation and then solve for x.

$$\begin{aligned} x^2 + \left(\frac{1}{x}\right)^2 &= 2 \\ x^2 + \frac{1}{x^2} &= 2 \end{aligned}$$

This leaves us with a rational equation. Make a note that $x \neq 0$ and multiply both sides by x^2.

$$\begin{aligned} x^2\left(x^2+\frac{1}{x^2}\right) &= 2\cdot x^2 \\ x^4+1 &= 2x^2 \\ x^4-2x^2+1 &= 0 \\ (x^2-1)(x^2-1) &= 0 \end{aligned}$$

At this point we can see that both factors are the same. Apply the zero product property.

$$\begin{aligned} x^2-1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

Back substitute into $y=\frac{1}{x}$ to find the corresponding y-values.

Using $x=-1$	Using $x=1$
$\begin{aligned} y &= \frac{1}{x} \\ &= \frac{1}{-1} \\ &= -1 \end{aligned}$	$\begin{aligned} y &= \frac{1}{x} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$

This leads to two solutions.

Answer: $(1, 1), (-1, -1)$

Try this! Solve: $\begin{cases} \frac{1}{x} + \frac{1}{y} = 4 \\ \frac{1}{x^2} + \frac{1}{y^2} = 40 \end{cases}$.

Answer: $\left(-\frac{1}{2}, \frac{1}{6}\right)\left(\frac{1}{6}, -\frac{1}{2}\right)$

(click to see video)

KEY TAKEAWAYS

- Use the substitution method to solve nonlinear systems.
- Streamline the solving process by using entire algebraic expressions in the substitution step to obtain a single equation with one variable.
- Understanding the geometric interpretation of the system can help in finding real solutions.

TOPIC EXERCISES

PART A: NONLINEAR SYSTEMS

Solve.

1. $\begin{cases} x^2 + y^2 = 10 \\ x + y = 4 \end{cases}$
2. $\begin{cases} x^2 + y^2 = 5 \\ x - y = -3 \end{cases}$
3. $\begin{cases} x^2 + y^2 = 30 \\ x - 3y = 0 \end{cases}$
4. $\begin{cases} x^2 + y^2 = 10 \\ 2x - y = 0 \end{cases}$
5. $\begin{cases} x^2 + y^2 = 18 \\ 2x - 2y = -12 \end{cases}$
6. $\begin{cases} (x - 4)^2 + y^2 = 25 \\ 4x - 3y = 16 \end{cases}$
7. $\begin{cases} 3x^2 + 2y^2 = 21 \\ 3x - y = 0 \end{cases}$
8. $\begin{cases} x^2 + 5y^2 = 36 \\ x - 2y = 0 \end{cases}$
9. $\begin{cases} 4x^2 + 9y^2 = 36 \\ 2x + 3y = 6 \end{cases}$
10. $\begin{cases} 4x^2 + y^2 = 4 \\ 2x + y = -2 \end{cases}$
11. $\begin{cases} 2x^2 + y^2 = 1 \\ x + y = 1 \end{cases}$

12. $$\begin{cases} 4x^2 + 3y^2 = 12 \\ 2x - y = 2 \end{cases}$$

13. $$\begin{cases} x^2 - 2y^2 = 35 \\ x - 3y = 0 \end{cases}$$

14. $$\begin{cases} 5x^2 - 7y^2 = 39 \\ 2x + 4y = 0 \end{cases}$$

15. $$\begin{cases} 9x^2 - 4y^2 = 36 \\ 3x + 2y = 0 \end{cases}$$

16. $$\begin{cases} x^2 + y^2 = 25 \\ x - 2y = -12 \end{cases}$$

17. $$\begin{cases} 2x^2 + 3y = 9 \\ 8x - 4y = 12 \end{cases}$$

18. $$\begin{cases} 2x - 4y^2 = 3 \\ 3x - 12y = 6 \end{cases}$$

19. $$\begin{cases} 4x^2 + 3y^2 = 12 \\ x - \frac{3}{2} = 0 \end{cases}$$

20. $$\begin{cases} 5x^2 + 4y^2 = 40 \\ y - 3 = 0 \end{cases}$$

21. The sum of the squares of two positive integers is 10. If the first integer is added to twice the second integer, the sum is 7. Find the integers.

22. The diagonal of a rectangle measures $\sqrt{5}$ units and has a perimeter equal to 6 units. Find the dimensions of the rectangle.

23. For what values of b will the following system have real solutions?
$$\begin{cases} x^2 + y^2 = 1 \\ y = x + b \end{cases}$$

24. For what values of m will the following system have real solutions?

$$\begin{cases} x^2 - y^2 = 1 \\ y = mx \end{cases}$$

Solve.

25. $$\begin{cases} x^2 + y^2 = 4 \\ y - x^2 = 2 \end{cases}$$

26. $$\begin{cases} x^2 + y^2 = 4 \\ y - x^2 = -2 \end{cases}$$

27. $$\begin{cases} x^2 + y^2 = 4 \\ y - x^2 = 3 \end{cases}$$

28. $$\begin{cases} x^2 + y^2 = 4 \\ 4y - x^2 = -4 \end{cases}$$

29. $$\begin{cases} x^2 + 3y^2 = 9 \\ y^2 - x = 3 \end{cases}$$

30. $$\begin{cases} x^2 + 3y^2 = 9 \\ x + y^2 = -4 \end{cases}$$

31. $$\begin{cases} 4x^2 - 3y^2 = 12 \\ x^2 + y^2 = 1 \end{cases}$$

32. $$\begin{cases} x^2 + y^2 = 1 \\ x^2 - y^2 = 1 \end{cases}$$

33. $$\begin{cases} x^2 + y^2 = 1 \\ 4y^2 - x^2 - 4y = 0 \end{cases}$$

34. $$\begin{cases} x^2 + y^2 = 4 \\ 2x^2 - y^2 + 4x = 0 \end{cases}$$

35. $$\begin{cases} 2(x-2)^2 + y^2 = 6 \\ (x-3)^2 + y^2 = 4 \end{cases}$$

36. $\begin{cases} x^2 + y^2 - 6y = 0 \\ 4x^2 + 5y^2 + 20y = 0 \end{cases}$

37. $\begin{cases} x^2 + 4y^2 = 25 \\ 4x^2 + y^2 = 40 \end{cases}$

38. $\begin{cases} x^2 - 2y^2 = -10 \\ 4x^2 + y^2 = 10 \end{cases}$

39. $\begin{cases} 2x^2 + y^2 = 14 \\ x^2 - (y-1)^2 = 6 \end{cases}$

40. $\begin{cases} 3x^2 - (y-2)^2 = 12 \\ x^2 + (y-2)^2 = 1 \end{cases}$

41. The difference of the squares of two positive integers is 12. The sum of the larger integer and the square of the smaller is equal to 8. Find the integers.

42. The difference between the length and width of a rectangle is 4 units and the diagonal measures 8 units. Find the dimensions of the rectangle. Round off to the nearest tenth.

43. The diagonal of a rectangle measures p units and has a perimeter equal to $2q$ units. Find the dimensions of the rectangle in terms of p and q.

44. The area of a rectangle is p square units and its perimeter is $2q$ units. Find the dimensions of the rectangle in terms of p and q.

Solve.

45. $\begin{cases} x^2 + y^2 = 26 \\ xy = 5 \end{cases}$

46. $\begin{cases} x^2 + y^2 = 10 \\ xy = 3 \end{cases}$

47. $\begin{cases} 2x^2 - 3y^2 = 5 \\ xy = 1 \end{cases}$

48. $\begin{cases} 3x^2 - 4y^2 = -11 \\ xy = 1 \end{cases}$

49. $\begin{cases} x^2 + y^2 = 2 \\ xy - 2 = 0 \end{cases}$

50. $\begin{cases} x^2 + y^2 = 1 \\ 2xy - 1 = 0 \end{cases}$

51. $\begin{cases} 4x - y^2 = 0 \\ xy = 2 \end{cases}$

52. $\begin{cases} 3y - x^2 = 0 \\ xy - 9 = 0 \end{cases}$

53. $\begin{cases} 2y - x^2 = 0 \\ xy - 1 = 0 \end{cases}$

54. $\begin{cases} x - y^2 = 0 \\ xy = 3 \end{cases}$

55. The diagonal of a rectangle measures $2\sqrt{10}$ units. If the area of the rectangle is 12 square units, find its dimensions.

56. The area of a rectangle is 48 square meters and the perimeter measures 32 meters. Find the dimensions of the rectangle.

57. The product of two positive integers is 72 and their sum is 18. Find the integers.

58. The sum of the squares of two positive integers is 52 and their product is 24. Find the integers.

Solve.

59. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 4 \\ \frac{1}{x} - \frac{1}{y} = 2 \end{cases}$

60. $\begin{cases} \frac{2}{x} - \frac{1}{y} = 5 \\ \frac{1}{x} + \frac{1}{y} = 2 \end{cases}$

61. $$\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} = 1 \\ \dfrac{3}{x} - \dfrac{1}{y} = 2 \end{cases}$$

62. $$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 6 \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = 20 \end{cases}$$

63. $$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 2 \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = 34 \end{cases}$$

64. $$\begin{cases} xy - 16 = 0 \\ 2x^2 - y = 0 \end{cases}$$

65. $$\begin{cases} x + y^2 = 4 \\ y = \sqrt{x} \end{cases}$$

66. $$\begin{cases} y^2 - (x-1)^2 = 1 \\ y = \sqrt{x} \end{cases}$$

67. $$\begin{cases} y = 2^x \\ y = 2^{2x} - 56 \end{cases}$$

68. $$\begin{cases} y = 3^{2x} - 72 \\ y - 3^x = 0 \end{cases}$$

69. $$\begin{cases} y = e^{4x} \\ y = e^{2x} + 6 \end{cases}$$

70. $$\begin{cases} y - e^{2x} = 0 \\ y - e^x = 0 \end{cases}$$

PART B: DISCUSSION BOARD

71. How many real solutions can be obtained from a system that consists of a circle and a hyperbola? Explain.

72. Make up your own nonlinear system, solve it, and provide the answer. Also, provide a graph and discuss the geometric interpretation of the solutions.

ANSWERS

1. $(1, 3)$, $(3, 1)$
3. $\left(-3\sqrt{3}, -\sqrt{3}\right)$, $\left(3\sqrt{3}, \sqrt{3}\right)$
5. $(-3, 3)$
7. $(-1, -3), (1, 3)$
9. $(0, 2), (3, 0)$
11. $(0, 1), \left(\frac{2}{3}, \frac{1}{3}\right)$
13. $\left(-3\sqrt{5}, -\sqrt{5}\right), \left(3\sqrt{5}, \sqrt{5}\right)$
15. Ø
17. $\left(\frac{-3+3\sqrt{5}}{2}, -6 + 3\sqrt{5}\right), \left(\frac{-3-3\sqrt{5}}{2}, -6 - 3\sqrt{5}\right)$
19. $\left(\frac{3}{2}, -1\right), \left(\frac{3}{2}, 1\right)$
21. 1, 3
23. $b \in \left[-\sqrt{2}, \sqrt{2}\right]$
25. $(0, 2)$
27. Ø
29. $(-3, 0), \left(0, -\sqrt{3}\right), \left(0, \sqrt{3}\right)$
31. Ø
33. $(0, 1), \left(-\frac{2\sqrt{5}}{5}, -\frac{1}{5}\right), \left(\frac{2\sqrt{5}}{5}, -\frac{1}{5}\right)$
35. $(3, -2), (3, 2)$
37. $(-3, -2), (-3, 2), (3, -2), (3, 2)$

39. $\left(-\sqrt{7}, 0\right), \left(\sqrt{7}, 0\right), \left(-\frac{\sqrt{55}}{3}, \frac{4}{3}\right), \left(\frac{\sqrt{55}}{3}, \frac{4}{3}\right)$

41. 2, 4

43. $\frac{q+\sqrt{2p^2-q^2}}{2}$ units by $\frac{q-\sqrt{2p^2-q^2}}{2}$ units

45. $(-5, -1), (5, 1), (-1, -5), (1, 5)$

47. $\left(-\sqrt{3}, -\frac{\sqrt{3}}{3}\right), \left(\sqrt{3}, \frac{\sqrt{3}}{3}\right)$

49. ∅

51. $(1, 2)$

53. $\left(\sqrt[3]{2}, \frac{\sqrt[3]{4}}{2}\right)$

55. 2 units by 6 units

57. 6, 12

59. $\left(\frac{1}{3}, 1\right)$

61. $\left(\frac{7}{5}, 7\right)$

63. $\left(-\frac{1}{3}, \frac{1}{5}\right), \left(\frac{1}{5}, -\frac{1}{3}\right)$

65. $\left(2, \sqrt{2}\right)$

67. $(3, 8)$

69. $\left(\frac{\ln 3}{2}, 9\right)$

71. Answer may vary

8.6 Review Exercises and Sample Exam

REVIEW EXERCISES

DISTANCE, MIDPOINT, AND THE PARABOLA

Calculate the distance and midpoint between the given two points.

1. $(0, 2)$ and $(-4, -1)$
2. $(6, 0)$ and $(-2, -6)$
3. $(-2, 4)$ and $(-6, -8)$
4. $\left(\frac{1}{2}, -1\right)$ and $\left(\frac{5}{2}, -\frac{1}{2}\right)$
5. $\left(0, -3\sqrt{2}\right)$ and $\left(\sqrt{5}, -4\sqrt{2}\right)$
6. $\left(-5\sqrt{3}, \sqrt{6}\right)$ and $\left(-3\sqrt{3}, \sqrt{6}\right)$

Determine the area of a circle whose diameter is defined by the given two points.

7. $(-3, 3)$ and $(3, -3)$
8. $(-2, -9)$ and $(-10, -15)$
9. $\left(\frac{2}{3}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{3}, \frac{3}{2}\right)$
10. $\left(2\sqrt{5}, -2\sqrt{2}\right)$ and $\left(0, -4\sqrt{2}\right)$

Rewrite in standard form and give the vertex.

11. $y = x^2 - 10x + 33$
12. $y = 2x^2 - 4x - 1$
13. $y = x^2 - 3x - 1$
14. $y = -x^2 - x - 2$
15. $x = y^2 + 10y + 10$

16. $x = 3y^2 + 12y + 7$
17. $x = -y^2 + 8y - 3$
18. $x = 5y^2 - 5y + 2$

Rewrite in standard form and graph. Be sure to find the vertex and all intercepts.

19. $y = x^2 - 20x + 75$
20. $y = -x^2 - 10x + 75$
21. $y = -2x^2 - 12x - 24$
22. $y = 4x^2 + 4x + 6$
23. $x = y^2 - 10y + 16$
24. $x = -y^2 + 4y + 12$
25. $x = -4y^2 + 12y$
26. $x = 9y^2 + 18y + 12$
27. $x = -4y^2 + 4y + 2$
28. $x = -y^2 - 5y + 2$

CIRCLES

Determine the center and radius given the equation of a circle in standard form.

29. $\left(x - 6\right)^2 + y^2 = 9$
30. $(x + 8)^2 + \left(y - 10\right)^2 = 1$
31. $x^2 + y^2 = 5$
32. $\left(x - \frac{3}{8}\right)^2 + \left(y + \frac{5}{2}\right)^2 = \frac{1}{2}$

Determine standard form for the equation of the circle:

33. Center $(-7, 2)$ with radius $r = 10$.

34. Center $\left(\frac{1}{3}, -1\right)$ with radius $r = \frac{2}{3}$.

35. Center $(0, -5)$ with radius $r = 2\sqrt{7}$.

36. Center $(1, 0)$ with radius $r = \frac{5\sqrt{3}}{2}$.

37. Circle whose diameter is defined by $(-4, 10)$ and $(-2, 8)$.

38. Circle whose diameter is defined by $(3, -6)$ and $(0, -4)$.

Find the *x*- and *y*-intercepts.

39. $(x-3)^2 + (y+5)^2 = 16$

40. $(x+5)^2 + (y-1)^2 = 4$

41. $x^2 + (y-2)^2 = 20$

42. $(x-3)^2 + (y+3)^2 = 8$

43. $x^2 + y^2 - 12y + 27 = 0$

44. $x^2 + y^2 - 4x + 2y + 1 = 0$

Graph.

45. $(x+8)^2 + (y-6)^2 = 4$

46. $(x-20)^2 + \left(y + \frac{15}{2}\right)^2 = \frac{225}{4}$

47. $x^2 + y^2 = 24$

48. $(x-1)^2 + y^2 = \frac{1}{4}$

49. $x^2 + (y-7)^2 = 27$

50. $(x+1)^2 + (y-1)^2 = 2$

Rewrite in standard form and graph.

51. $x^2 + y^2 - 6x + 4y - 3 = 0$
52. $x^2 + y^2 + 8x - 10y + 16 = 0$
53. $2x^2 + 2y^2 - 2x - 6y - 3 = 0$
54. $4x^2 + 4y^2 + 8y + 1 = 0$
55. $x^2 + y^2 - 5x + y - \frac{1}{2} = 0$
56. $x^2 + y^2 + 12x - 8y = 0$

ELLIPSES

Given the equation of an ellipse in standard form, determine its center, orientation, major radius, and minor radius.

57. $\frac{(x+12)^2}{16} + \frac{(y-10)^2}{4} = 1$
58. $\frac{(x+3)^2}{3} + \frac{y^2}{25} = 1$
59. $x^2 + \frac{(y-5)^2}{12} = 1$
60. $\frac{(x-8)^2}{5} + \frac{(y+8)}{18} = 1$

Determine the standard form for the equation of the ellipse given the following information.

61. Center $(0, -4)$ with $a = 3$ and $b = 4$.
62. Center $(3, 8)$ with $a = 1$ and $b = \sqrt{7}$.
63. Center $(0, 0)$ with $a = 5$ and $b = \sqrt{2}$.
64. Center $(-10, -30)$ with $a = 10$ and $b = 1$.

Find the *x*- and *y*-intercepts.

65. $\frac{(x+2)^2}{4} + \frac{y^2}{9} = 1$

66. $\frac{(x-1)^2}{2} + \frac{(y+1)^2}{3} = 1$

67. $5x^2 + 2y^2 = 20$

68. $5(x-3)^2 + 6y^2 = 120$

Graph.

69. $\frac{(x-10)^2}{25} + \frac{(y+5)^2}{4} = 1$

70. $\frac{(x+6)^2}{9} + \frac{(y-8)^2}{36} = 1$

71. $\frac{\left(x-\frac{3}{2}\right)^2}{4} + \left(y - \frac{7}{2}\right)^2 = 1$

72. $\left(x - \frac{2}{3}\right)^2 + \frac{y^2}{4} = 1$

73. $\frac{x^2}{2} + \frac{y^2}{5} = 1$

74. $\frac{(x+2)^2}{8} + \frac{(y-3)^2}{12} = 1$

Rewrite in standard form and graph.

75. $4x^2 + 9y^2 - 8x + 90y + 193 = 0$

76. $9x^2 + 4y^2 + 108x - 80y + 580 = 0$

77. $x^2 + 9y^2 + 6x + 108y + 324 = 0$

78. $25x^2 + y^2 - 350x - 8y + 1{,}216 = 0$

79. $8x^2 + 12y^2 - 16x - 36y - 13 = 0$

80. $10x^2 + 2y^2 - 50x + 14y + 7 = 0$

HYPERBOLAS

Given the equation of a hyperbola in standard form, determine its center, which way the graph opens, and the vertices.

81. $\frac{(x-10)^2}{4} - \frac{(y+5)^2}{16} = 1$

82. $\frac{(x+7)^2}{2} - \frac{(y-8)^2}{8} = 1$

83. $\frac{(y-20)^2}{3} - (x-15)^2 = 1$

84. $3y^2 - 12(x-1)^2 = 36$

Determine the standard form for the equation of the hyperbola.

85. Center $(-25, 10)$, $a = 3, b = \sqrt{5}$, opens up and down.

86. Center $(9, -12)$, $a = 5\sqrt{3}, b = 7$, opens left and right.

87. Center $(-4, 0)$, $a = 1, b = 6$, opens left and right.

88. Center $(-2, -3)$, $a = 10\sqrt{2}, b = 2\sqrt{3}$, opens up and down.

Find the *x*- and *y*-intercepts.

89. $\frac{(x-1)^2}{4} - \frac{(y+3)^2}{9} = 1$

90. $\frac{(x+4)^2}{8} - \frac{(y-2)^2}{12} = 1$

91. $4(y-2)^2 - x^2 = 16$

92. $6(y+1)^2 - 3(x-1)^2 = 18$

Graph.

93. $\frac{(x-10)^2}{25} - \frac{(y+5)^2}{100} = 1$

94. $\frac{(x-4)^2}{4} - \frac{(y-8)^2}{16} = 1$

95. $\frac{(y-3)^2}{9} - \frac{(x-6)^2}{81} = 1$

96. $\frac{(y+1)^2}{4} - \frac{(x+1)^2}{25} = 1$

97. $\frac{y^2}{27} - \frac{(x-3)^2}{9} = 1$

98. $\frac{x^2}{2} - \frac{y^2}{3} = 1$

Rewrite in standard form and graph.

99. $4x^2 - 9y^2 - 8x - 90y - 257 = 0$

100. $9x^2 - y^2 - 108x + 16y + 224 = 0$

101. $25y^2 - 2x^2 - 100y + 50 = 0$

102. $3y^2 - x^2 - 2x - 10 = 0$

103. $8y^2 - 12x^2 + 24y - 12x - 33 = 0$

104. $4y^2 - 4x^2 - 16y - 28x - 37 = 0$

Identify the conic sections and rewrite in standard form.

105. $x^2 + y^2 - 2x - 8y + 16 = 0$

106. $x^2 + 2y^2 + 4x - 24y + 74 = 0$

107. $x^2 - y^2 - 6x - 4y + 3 = 0$

108. $x^2 + y - 10x + 22 = 0$

109. $x^2 + 12y^2 - 12x + 24 = 0$

110. $x^2 + y^2 + 10y + 22 = 0$

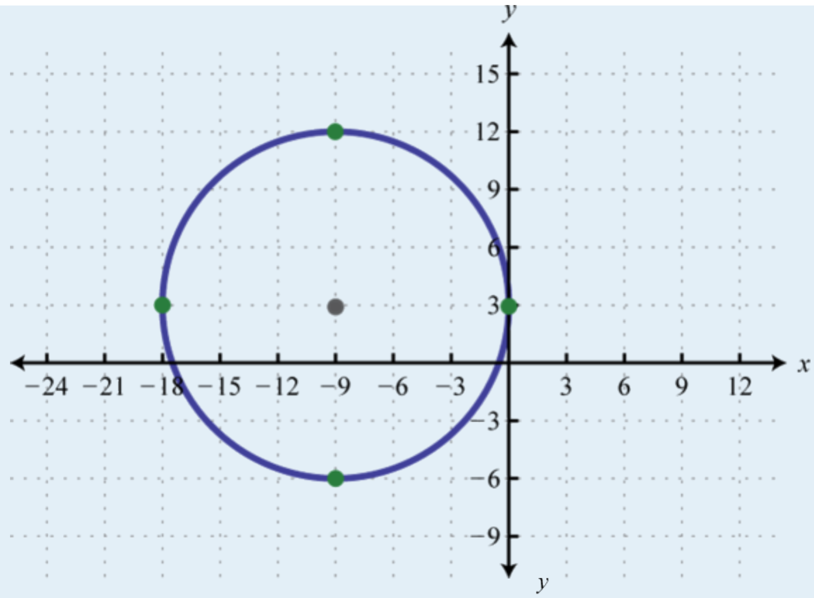
111. $4y^2 - 20x^2 + 16y + 20x - 9 = 0$

112. $16x - 16y^2 + 24y - 25 = 0$

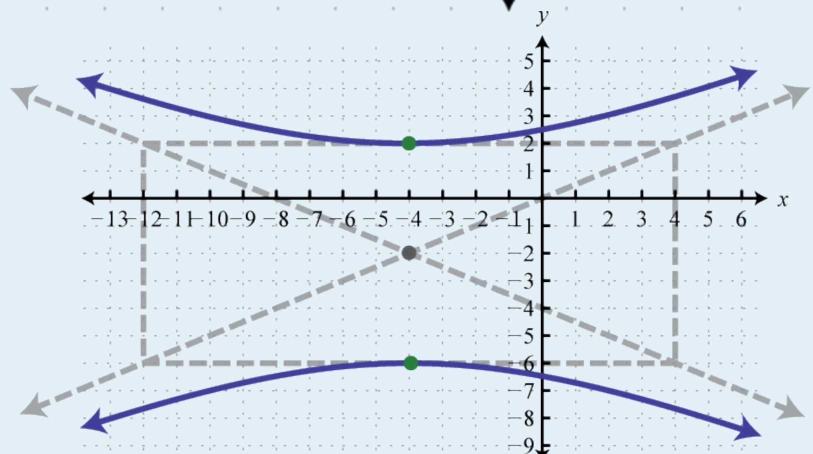
113. $9x^2 - 9y^2 - 6x - 18y - 17 = 0$

114. $4x^2 + 4y^2 + 4x - 8y + 1 = 0$

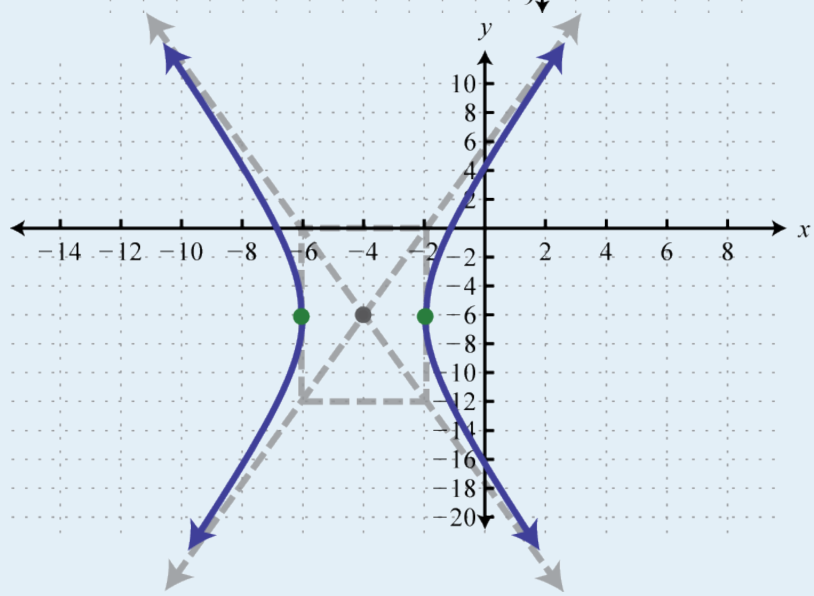
Given the graph, write the equation in general form.


y
15
12
9
6
3
−3
−6
−9
x
−24 −21 −18 −15 −12 −9 −6 −3 3 6 9 12

115.


y
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−7
−8
−9
x
−13 −12 −11 −10 −9 −8 −7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6

116.


y
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8
6
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−2
−4
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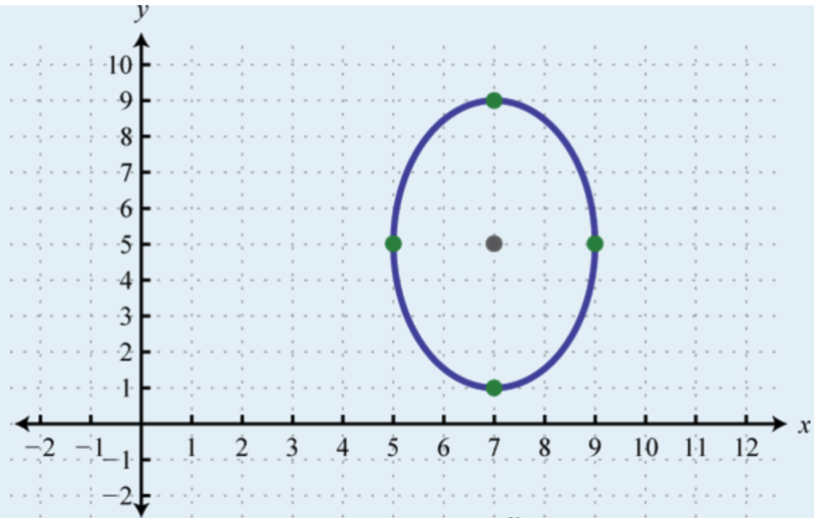
−10
−12
−14
−16
−18
−20
x
−14 −12 −10 −8 −6 −4 −2 2 4 6 8

117.



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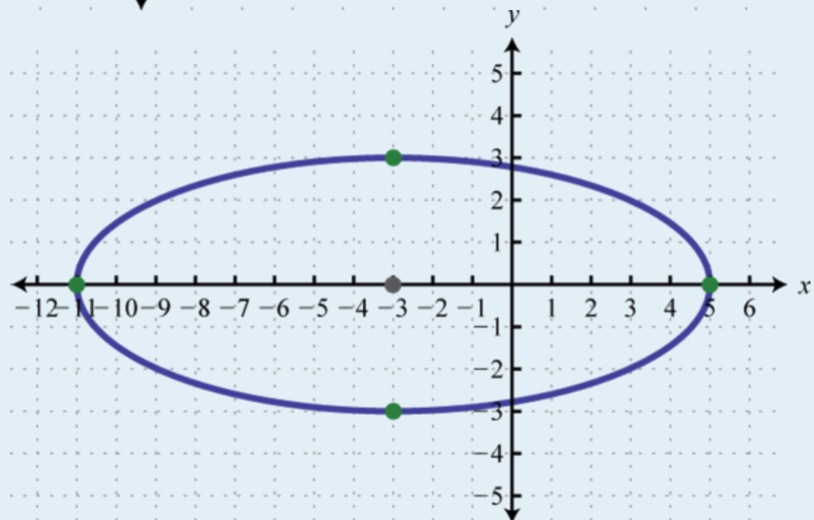

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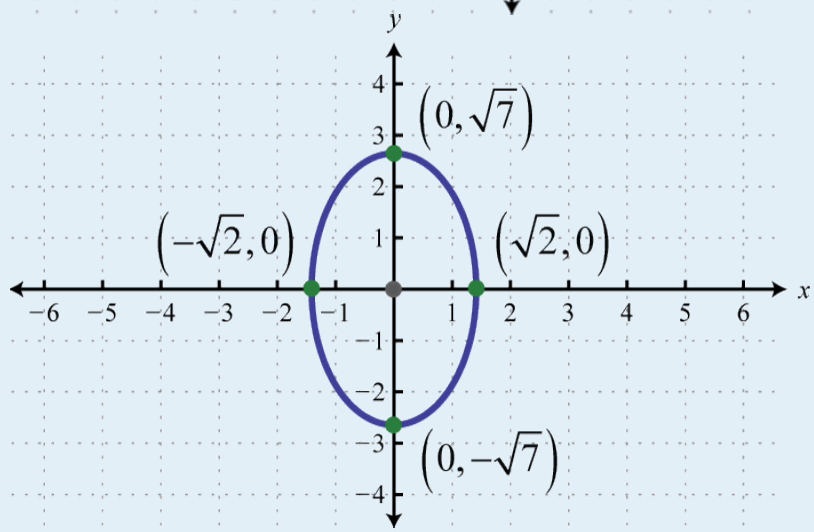


120.

SOLVING NONLINEAR SYSTEMS

Solve.

121. $\begin{cases} x^2 + y^2 = 8 \\ x - y = 4 \end{cases}$

122. $\begin{cases} x^2 + y^2 = 1 \\ x + 2y = 1 \end{cases}$

123. $\begin{cases} x^2 + 3y^2 = 4 \\ 2x - y = 1 \end{cases}$

124. $\begin{cases} 2x^2 + y^2 = 5 \\ x + y = 3 \end{cases}$

125. $\begin{cases} 3x^2 - 2y^2 = 1 \\ x - y = 2 \end{cases}$

126. $\begin{cases} x^2 - 3y^2 = 10 \\ x - 2y = 1 \end{cases}$

127. $\begin{cases} 2x^2 + y^2 = 11 \\ 4x + y^2 = 5 \end{cases}$

128. $\begin{cases} x^2 + 4y^2 = 1 \\ 2x^2 + 4y = 5 \end{cases}$

129. $\begin{cases} 5x^2 - y^2 = 10 \\ x^2 + y = 2 \end{cases}$

130. $\begin{cases} 2x^2 + y^2 = 1 \\ 2x - 4y^2 = -3 \end{cases}$

131. $\begin{cases} x^2 + 4y^2 = 10 \\ xy = 2 \end{cases}$

132. $\begin{cases} y + x^2 = 0 \\ xy - 8 = 0 \end{cases}$

133. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 10 \\ \dfrac{1}{x} - \dfrac{1}{y} = 6 \end{cases}$

134. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 1 \\ y - x = 2 \end{cases}$

135. $\begin{cases} x - 2y^2 = 3 \\ y = \sqrt{x - 4} \end{cases}$

136. $\begin{cases} (x - 1)^2 + y^2 = 1 \\ y - \sqrt{x} = 0 \end{cases}$

ANSWERS

1. Distance: 5 units; midpoint: $\left(-2, \frac{1}{2}\right)$

3. Distance: $4\sqrt{10}$ units; midpoint: $(-4, -2)$

5. Distance: $\sqrt{7}$ units; midpoint: $\left(\frac{\sqrt{5}}{2}, -\frac{7\sqrt{2}}{2}\right)$

7. 18π square units

9. $\frac{5\pi}{4}$ square units

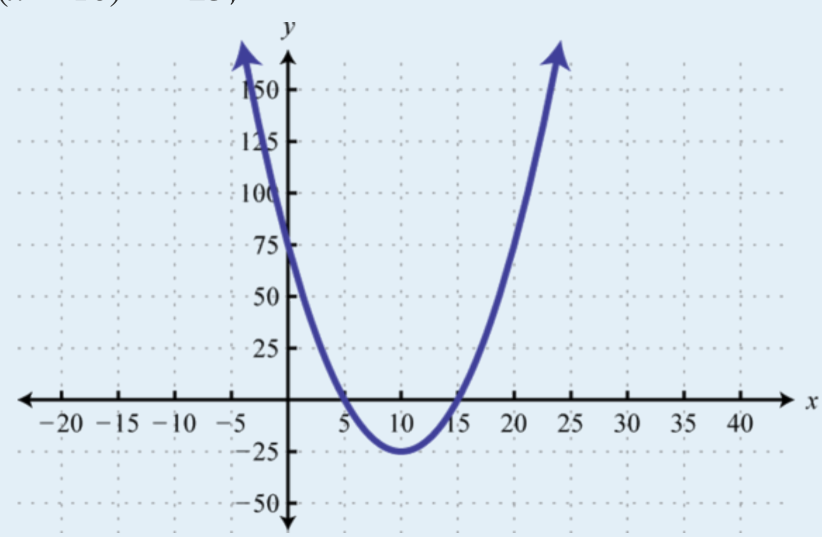
11. $y = (x-5)^2 + 8$; vertex: $(5, 8)$

13. $y = \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$; vertex: $\left(\frac{3}{2}, -\frac{13}{4}\right)$

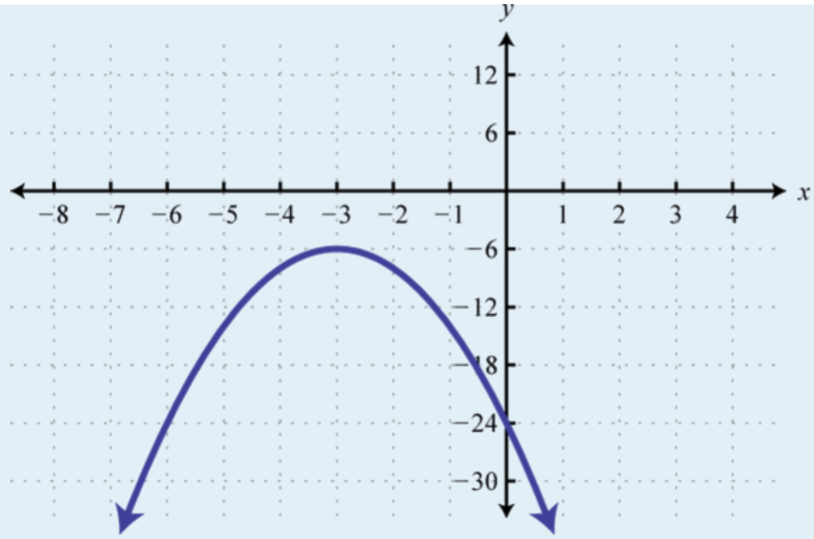
15. $x = (y+5)^2 - 15$; vertex: $(-15, -5)$

17. $x = -(y-4)^2 + 13$; vertex: $(13, 4)$

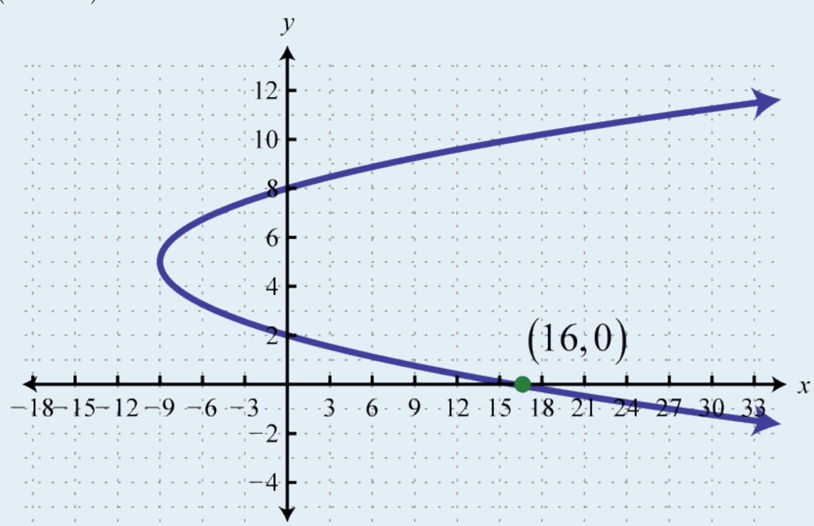
19. $y = (x-10)^2 - 25$;



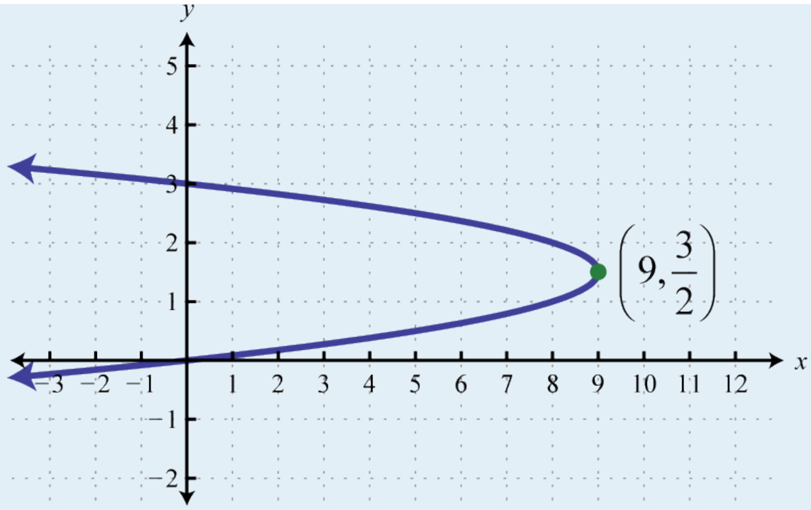
21. $y = -2(x+3)^2 - 6$;



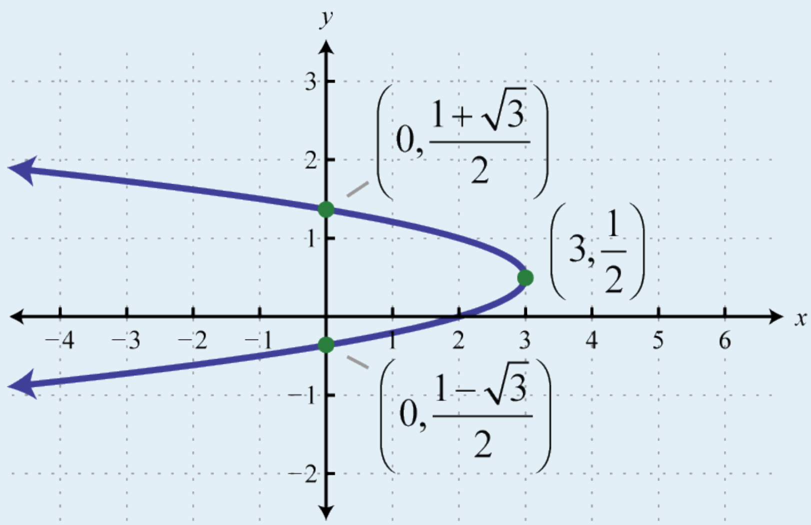
23. $x = (y-5)^2 - 9$;



25. $x = -4\left(y - \frac{3}{2}\right)^2 + 9$;



27. $x = -4\left(y - \frac{1}{2}\right)^2 + 3;$



29. Center: $(6, 0)$; radius: $r = 3$

31. Center: $(0, 0)$; radius: $r = \sqrt{5}$

33. $(x + 7)^2 + (y - 2)^2 = 100$

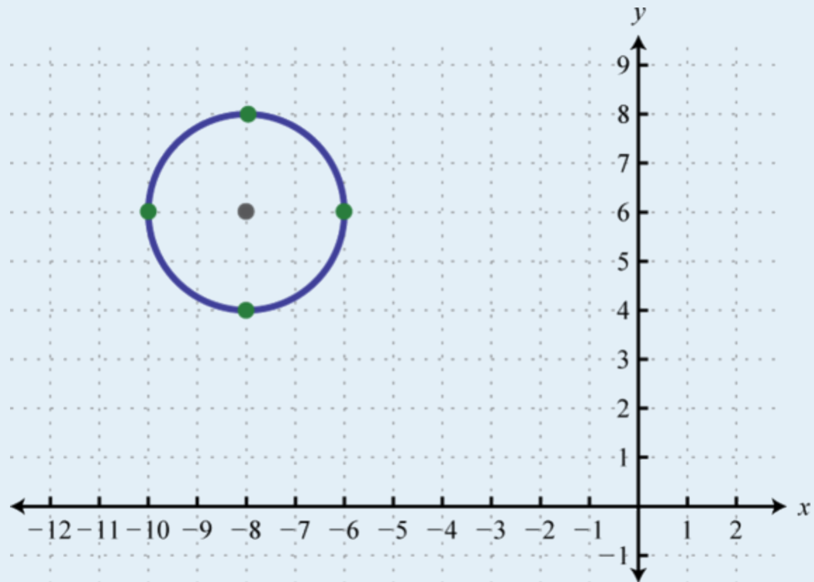
35. $x^2 + (y + 5)^2 = 28$

37. $(x + 3)^2 + (y - 9)^2 = 2$

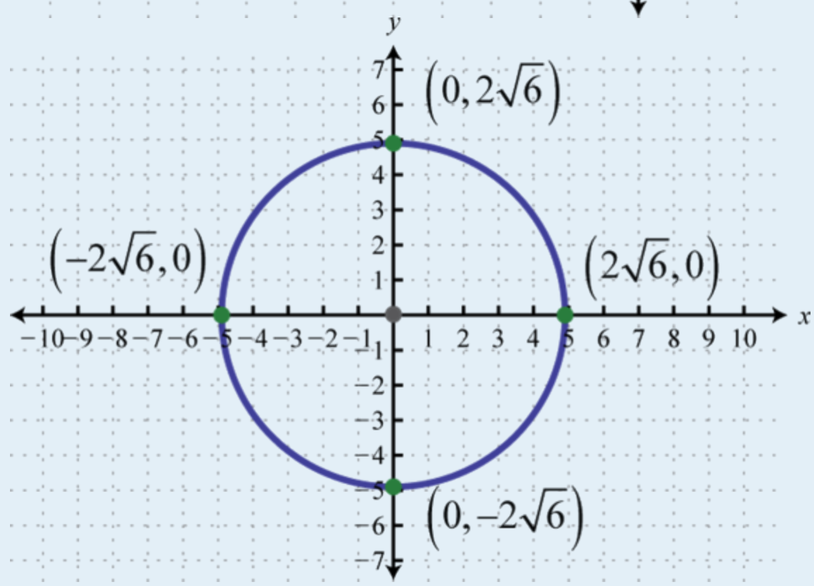
39. x-intercepts: none; y-intercepts: $\left(0, -5 \pm \sqrt{7}\right)$

41. x-intercepts: $(\pm 4, 0)$; y-intercepts: $\left(0, 2 \pm 2\sqrt{5}\right)$

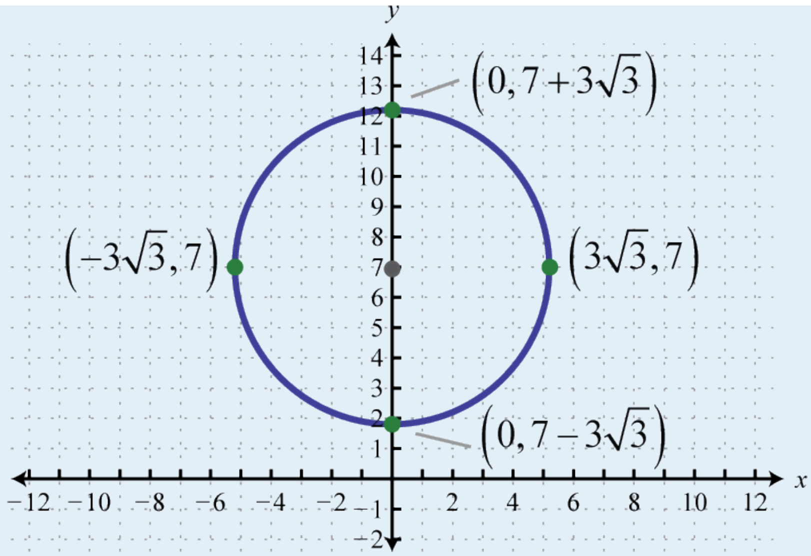
43. x-intercepts: none; y-intercepts: $(0, 3), (0, 9)$



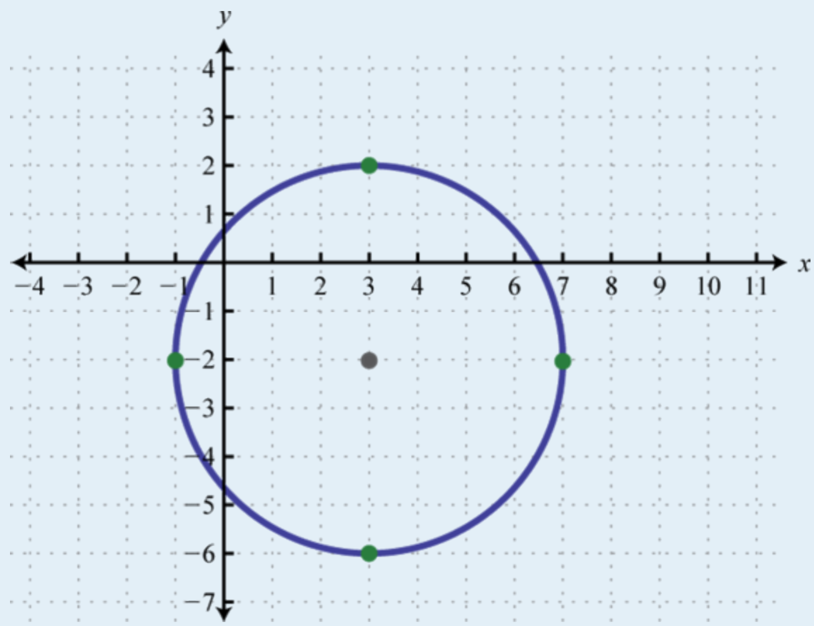
45.



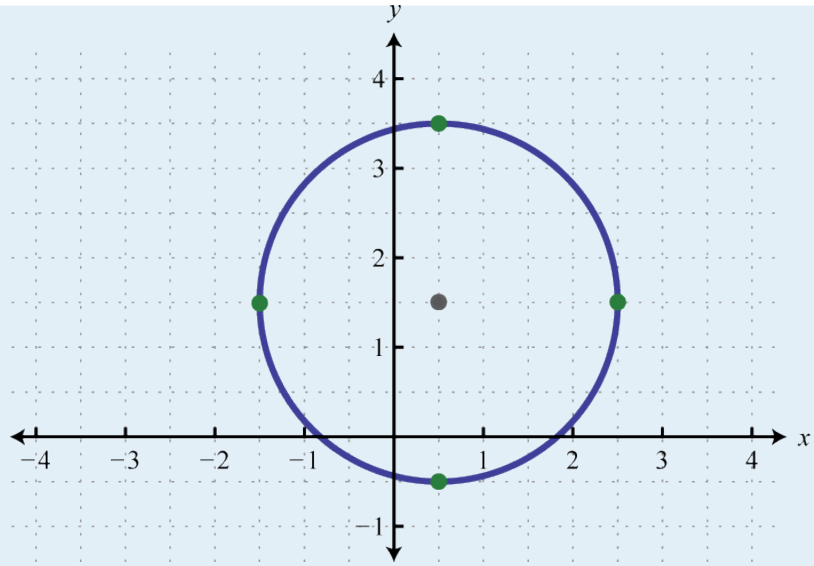
47.

49. 

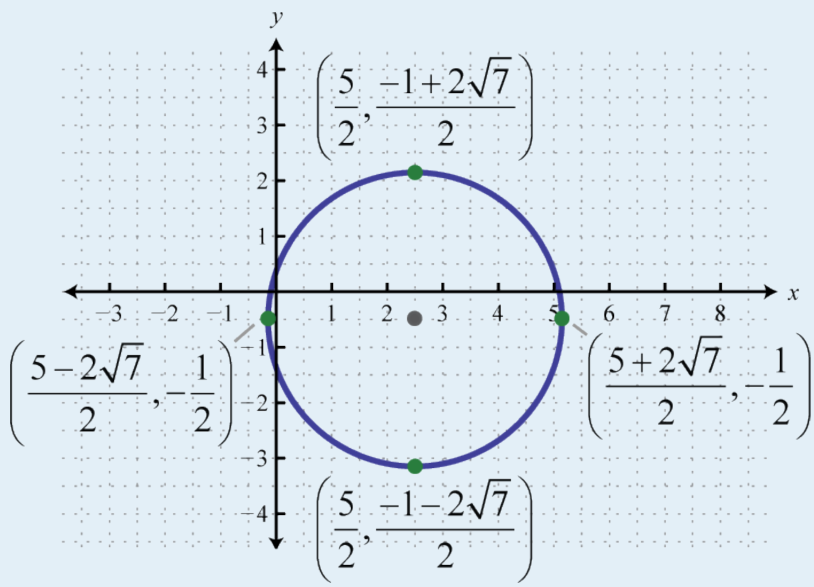
51. $(x-3)^2+\left(y+2\right)^2=16$;



53. $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=4$;



55. $\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 7$;



57. Center: $(-12, 10)$; orientation: horizontal; major radius: 4 units; minor radius: 2 units

59. Center: $(0, 5)$; orientation: vertical; major radius: $2\sqrt{3}$ units; minor radius: 1 unit

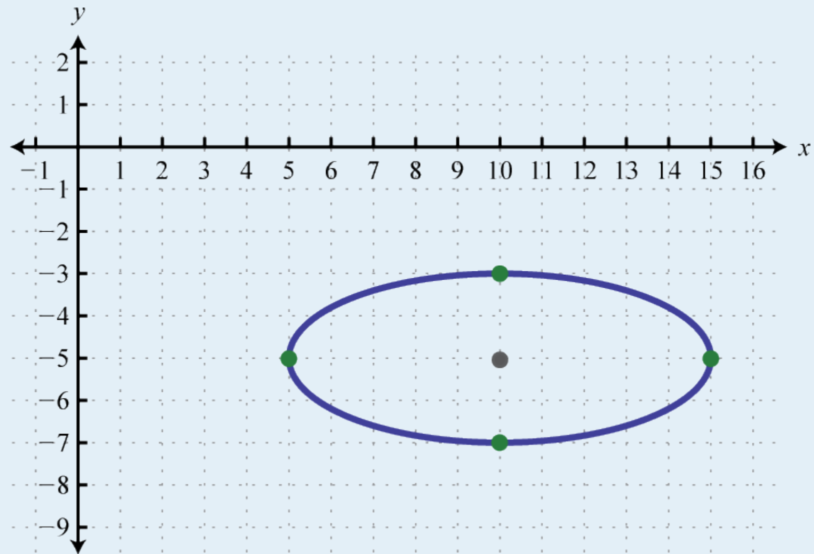
61. $\frac{x^2}{9} + \frac{(y+4)^2}{16} = 1$

63. $\frac{x^2}{25} + \frac{y^2}{2} = 1$

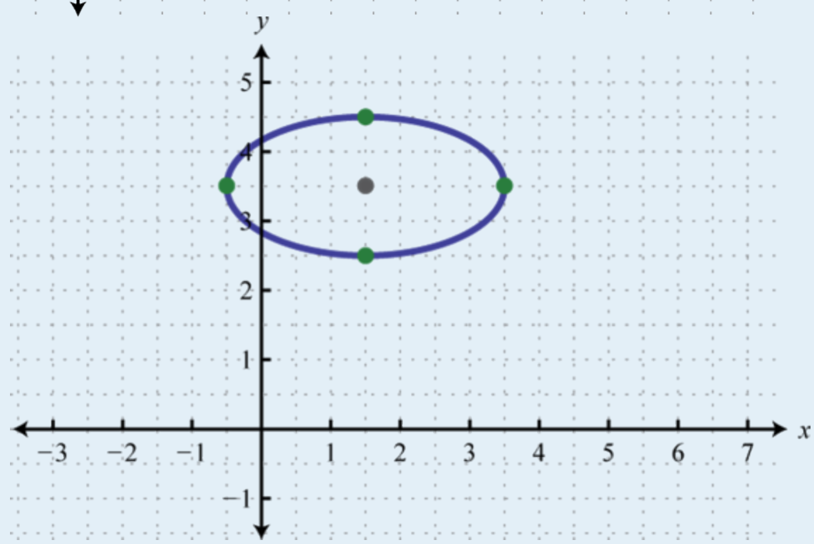
65. x-intercepts: $(-4, 0), (0, 0)$; y-intercepts: $(0, 0)$

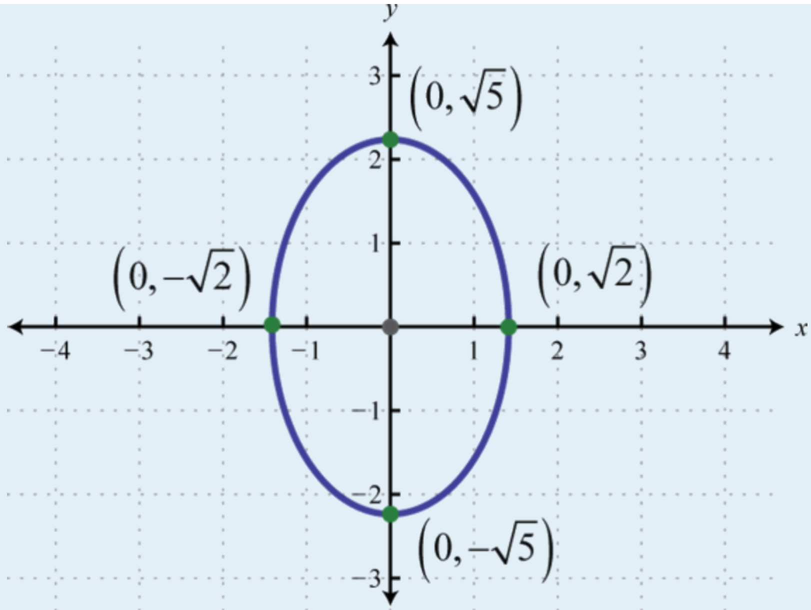
67. x-intercepts: $(\pm 2, 0)$; y-intercepts: $\left(0, \pm\sqrt{10}\right)$

69.



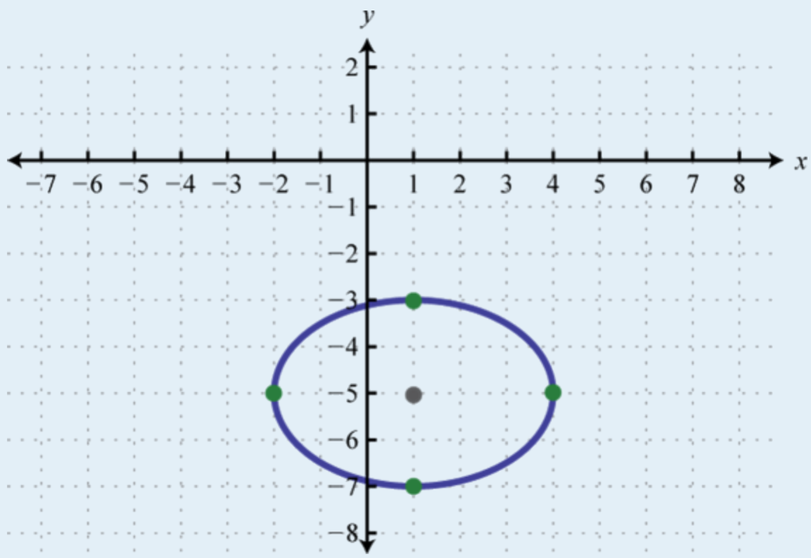
71.



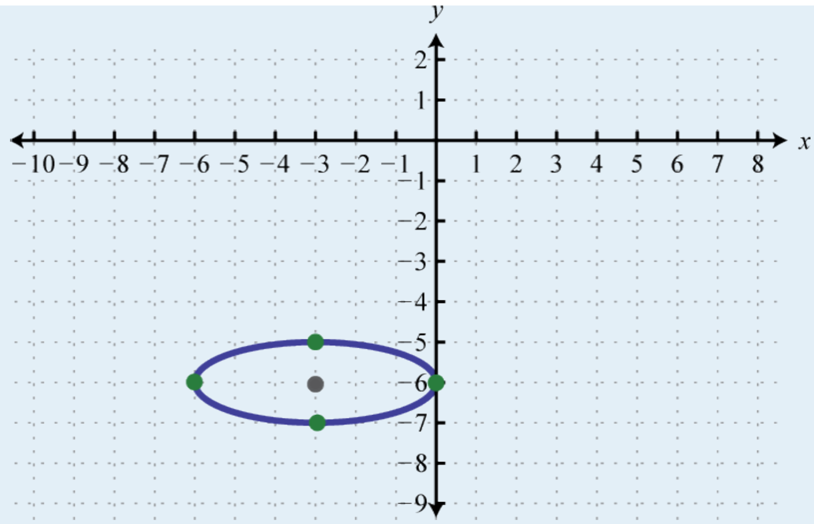


73.

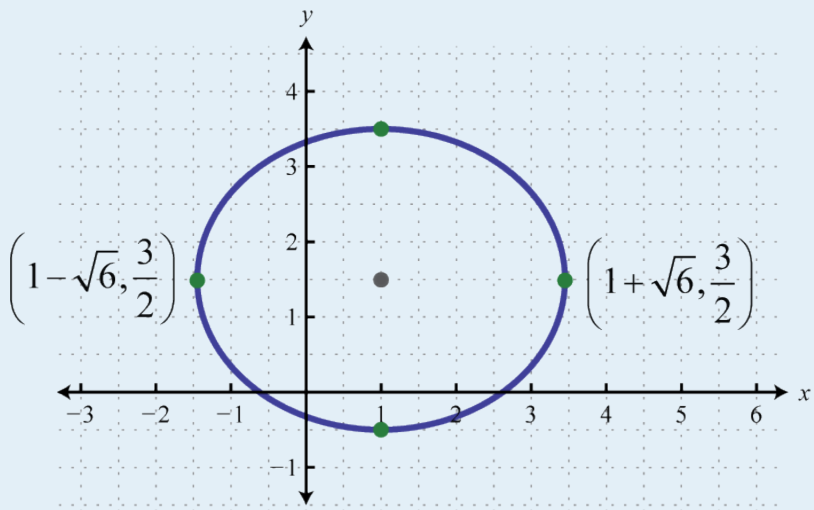
75. $\frac{(x-1)^2}{9} + \frac{(y+5)^2}{4} = 1;$



77. $\frac{(x+3)^2}{9} + (y+6)^2 = 1;$



79. $\frac{(x-1)^2}{6} + \frac{\left(y-\frac{3}{2}\right)^2}{4} = 1$;



81. Center: $(10, -5)$; opens left and right; vertices: $(8, -5)$, $(12, -5)$

83. Center: $(15, 20)$; opens upward and downward; vertices: $\left(15, 20 - \sqrt{3}\right)$, $\left(15, 20 + \sqrt{3}\right)$

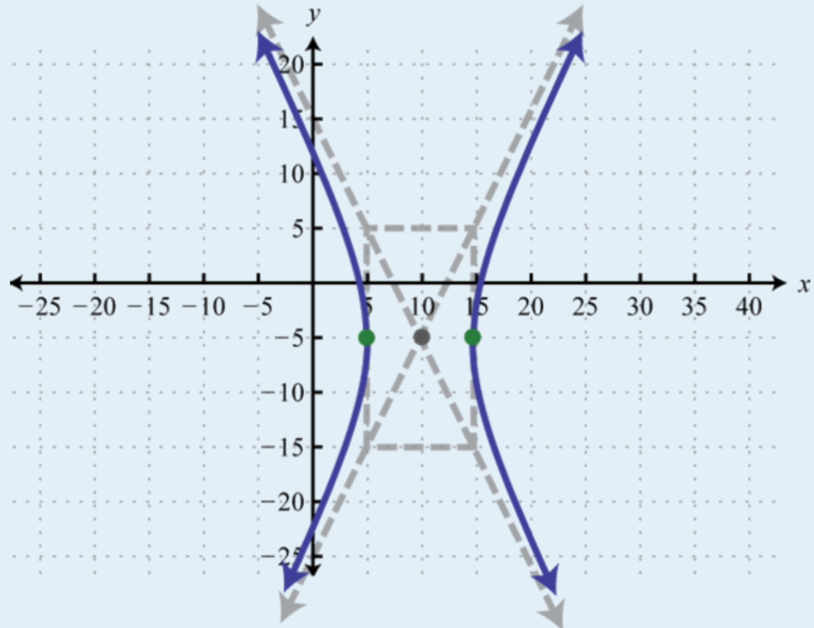
85. $\frac{(y-10)^2}{5} - \frac{(x+25)^2}{9} = 1$

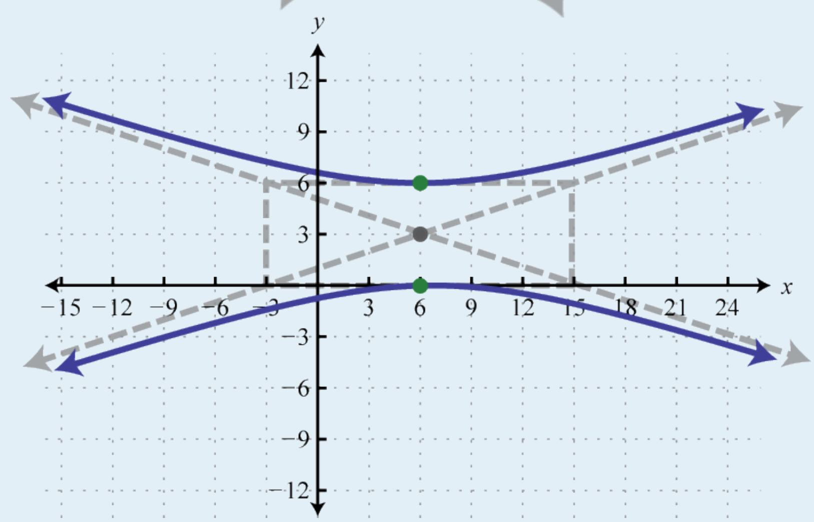
87. $(x+4)^2 - \frac{y^2}{36} = 1$

89. x-intercepts: $\left(1 \pm 2\sqrt{2}, 0\right)$; y-intercepts: none

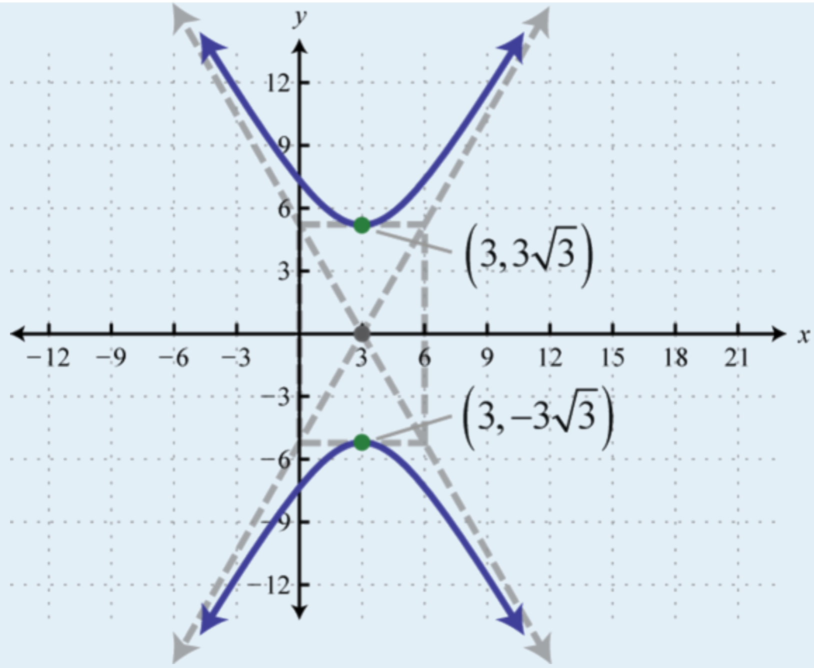
91. x-intercepts: $(0, 0)$; y-intercepts: $(0, 0), (0, 4)$

93.



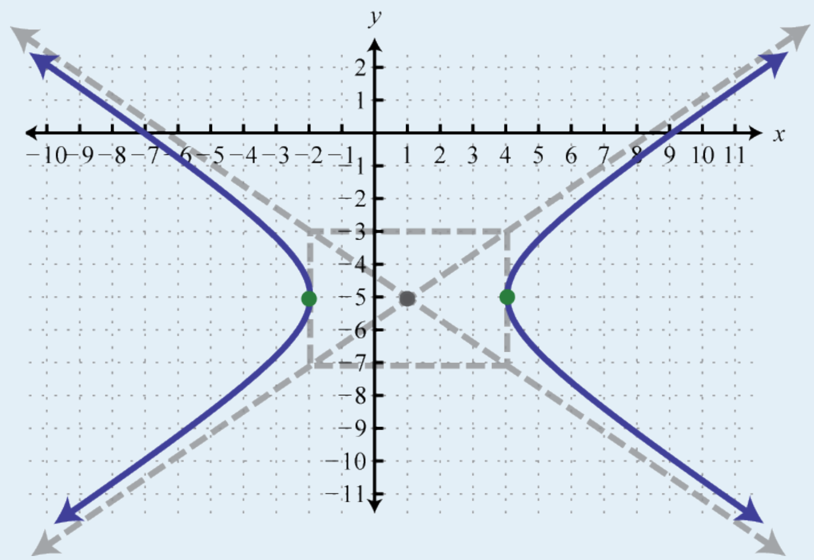


95.

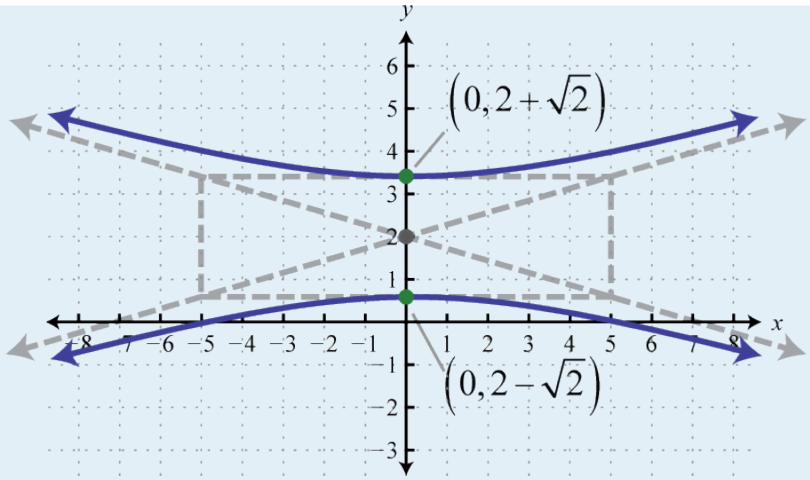


97.

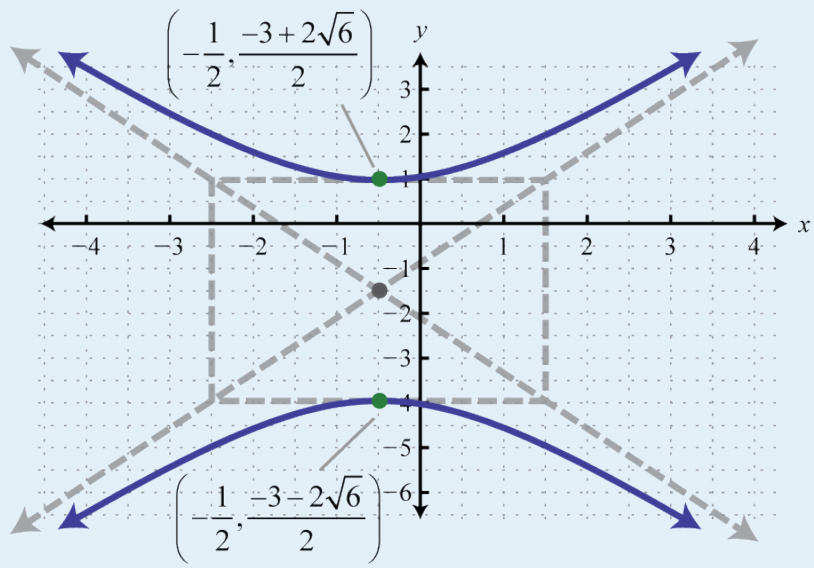
99. $\frac{(x-1)^2}{9} - \frac{(y+5)^2}{4} = 1;$



101. $\frac{(y-2)^2}{2} - \frac{x^2}{25} = 1;$



103. $\frac{\left(y+\frac{3}{2}\right)^2}{6} - \frac{\left(x+\frac{1}{2}\right)^2}{4} = 1$;



105. Circle; $(x-1)^2 + \left(y-4\right)^2 = 1$

107. Hyperbola; $\frac{(x-3)^2}{2} - \frac{\left(y+2\right)^2}{2} = 1$

109. Ellipse; $\frac{\left(x-6\right)^2}{12} + y^2 = 1$

111. Hyperbola; $\frac{\left(y+2\right)^2}{5} - \left(x-\frac{1}{2}\right)^2 = 1$

113. Hyperbola; $\left(x-\frac{1}{3}\right)^2 - \left(y+1\right)^2 = 1$

115. $x^2 + y^2 + 18x - 6y + 9 = 0$

117. $9x^2 - y^2 + 72x - 12y + 72 = 0$

119. $9x^2 + 64y^2 + 54x - 495 = 0$

121. $(2, -2)$

123. $\left(-\frac{1}{13}, -\frac{15}{13}\right), (1, 1)$

125. $(-9, -11), (1, -1)$

127. $(-1, -3), (-1, 3)$

129. $\left(-\sqrt{2}, 0\right), \left(\sqrt{2}, 0\right), \left(-\sqrt{7}, -5\right), \left(\sqrt{7}, -5\right)$

131. $\left(\sqrt{2}, \sqrt{2}\right), \left(-\sqrt{2}, -\sqrt{2}\right), \left(2\sqrt{2}, \frac{\sqrt{2}}{2}\right), \left(-2\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$

133. $\left(\frac{1}{8}, \frac{1}{2}\right)$

135. $(5, 1)$

SAMPLE EXAM

1. Given two points $(-4, -6)$ and $(2, -8)$:
   a. Calculate the distance between them.
   b. Find the midpoint between them.
2. Determine the area of a circle whose diameter is defined by the points $(4, -3)$ and $(-1, 2)$.

Rewrite in standard form and graph. Find the vertex and all intercepts if any.

3. $y = -x^2 + 6x - 5$
4. $x = 2y^2 + 4y - 6$
5. $x = -3y^2 + 3y + 1$
6. Find the equation of a circle in standard form with center $(-6, 3)$ and radius $2\sqrt{5}$ units.

Sketch the graph of the conic section given its equation in standard form.

7. $(x-4)^2 + (y+1)^2 = 45$
8. $\frac{(x+3)^2}{4} + \frac{y^2}{9} = 1$
9. $\frac{y^2}{3} - \frac{x^2}{9} = 1$
10. $\frac{x^2}{16} - (y-2)^2 = 1$

Rewrite in standard form and graph.

11. $9x^2 + 4y^2 - 144x + 16y + 556 = 0$
12. $x - y^2 + 6y + 7 = 0$
13. $x^2 + y^2 + 20x - 20y + 100 = 0$
14. $4y^2 - x^2 + 40y - 30x - 225 = 0$

Find the *x*- and *y*-intercepts.

15. $x = -2\left(y - 4\right)^2 + 9$

16. $\frac{(y-1)^2}{12} - (x+1)^2 = 1$

Solve.

17. $\begin{cases} x + y = 2 \\ y = -x^2 + 4 \end{cases}$

18. $\begin{cases} y - x^2 = -3 \\ x^2 + y^2 = 9 \end{cases}$

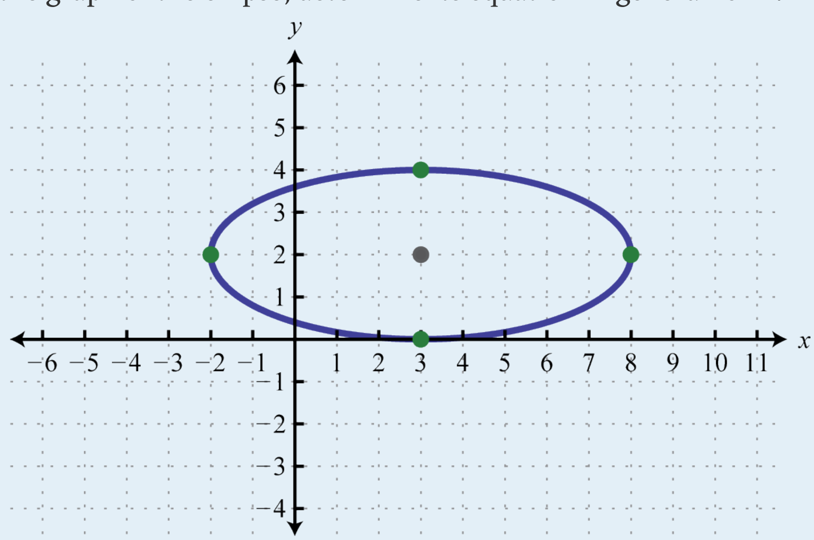
19. $\begin{cases} 2x - y = 1 \\ (x+1)^2 + 2y^2 = 1 \end{cases}$

20. $\begin{cases} x^2 + y^2 = 6 \\ xy = 3 \end{cases}$

21. Find the equation of an ellipse in standard form with vertices $\left(-3, -5\right)$ and $\left(5, -5\right)$ and a minor radius 2 units in length.

22. Find the equation of a hyperbola in standard form opening left and right with vertices $\left(\pm\sqrt{5}, 0\right)$ and a conjugate axis that measures 10 units.

23. Given the graph of the ellipse, determine its equation in general form.



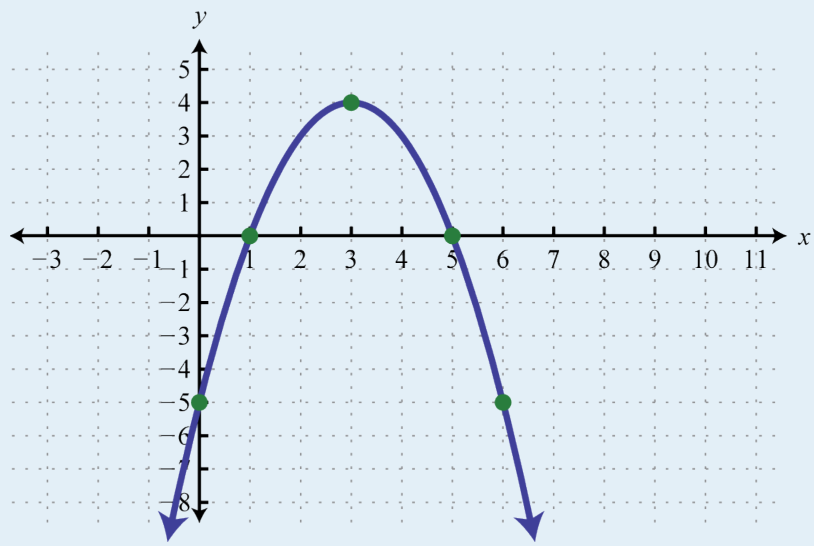
24. A rectangular deck has an area of 80 square feet and a perimeter that measures 36 feet. Find the dimensions of the deck.

25. The diagonal of a rectangle measures $2\sqrt{13}$ centimeters and the perimeter measures 20 centimeters. Find the dimensions of the rectangle.

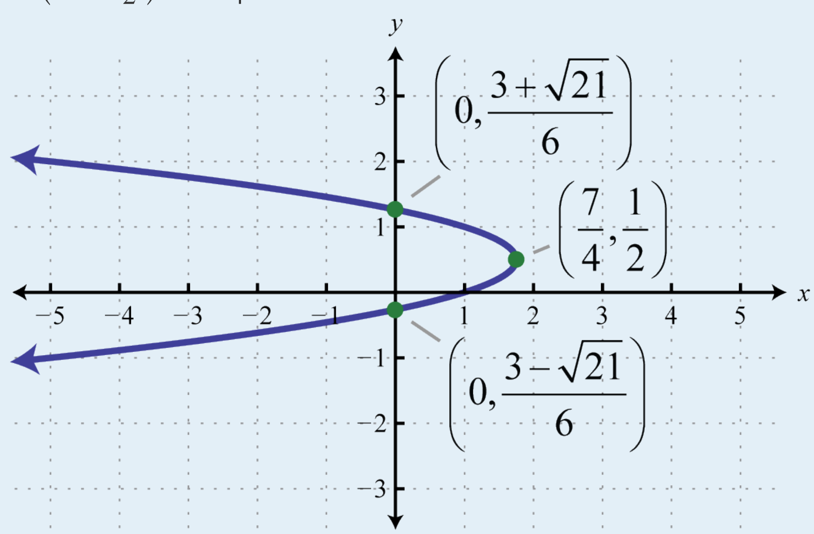
ANSWERS

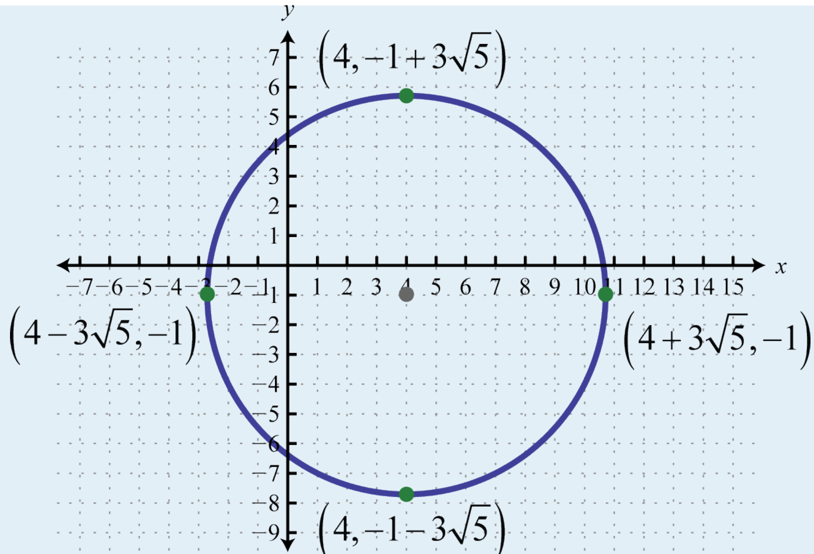
1. a. $2\sqrt{10}$ units;
   b. $(-1, -7)$

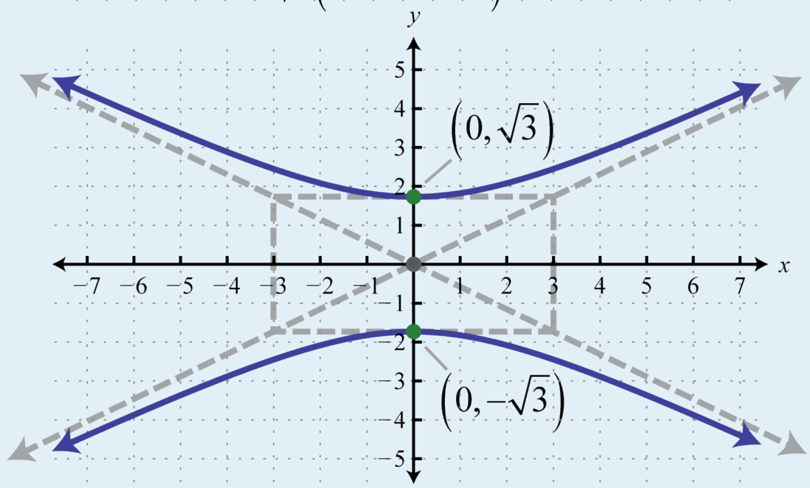
3. $y = -(x-3)^2 + 4$;



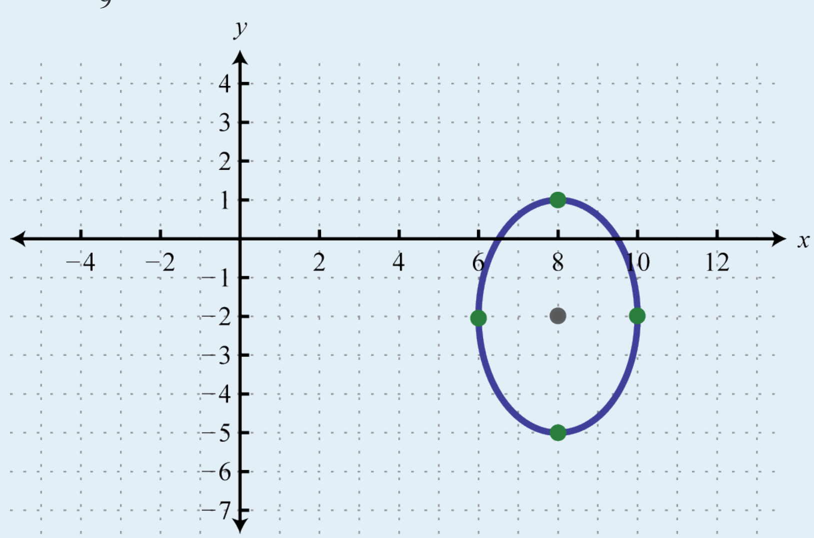
5. $x = -3\left(y - \frac{1}{2}\right)^2 + \frac{7}{4}$;



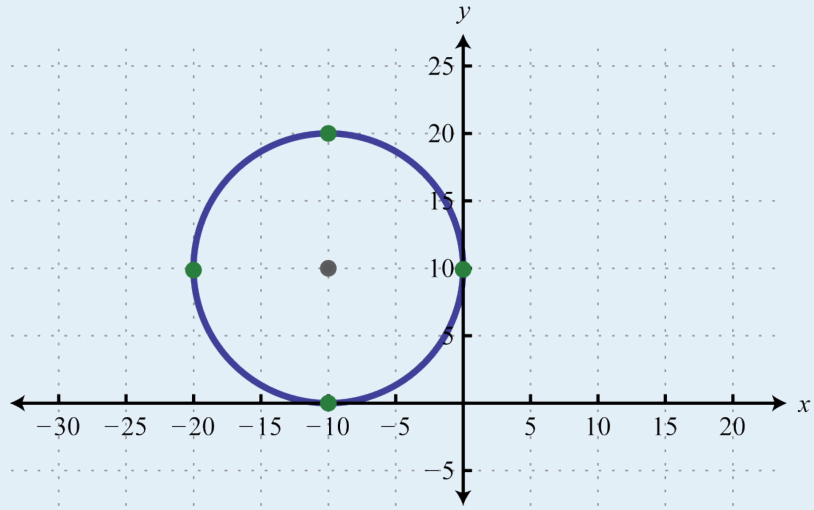
7. 

9. 

11. $\frac{(x-8)^2}{4}+\frac{(y+2)^2}{9}=1$;



13. $(x + 10)^2 + \left(y - 10\right)^2 = 100$;



15. x-intercept: $(-23, 0)$; y-intercepts: $\left(0, \frac{8 \pm 3\sqrt{2}}{2}\right)$

17. $(-1, 3), (2, 0)$

19. Ø

21. $\frac{(x-1)^2}{16} + \frac{\left(y+5\right)^2}{4} = 1$

23. $4x^2 + 25y^2 - 24x - 100y + 36 = 0$

25. 6 centimeters by 4 centimeters